CHASING RENORMALONS IN ONE DIMENSION

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Transseries in QFT

Asymptotic series in QFT

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If φ is *Borel summable* and we recover the "true" function $\varphi(z)$ from the Borel sum

$$s(\varphi)(z) = \int_0^\infty \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)d\zeta. \tag{1.3}$$

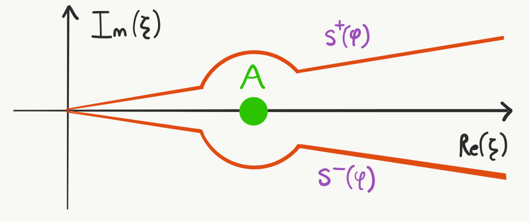
Ambiguity strikes back

If we Borel transform the example from before with $A > 0$

$$F_p(g) \sim \sum_{k\geq 0}^{\infty} (A^{-k} k!) g^k \Rightarrow \widehat{F}(\zeta) = \frac{1}{1 - \zeta/A} \tag{1.4}$$

There's a pole on $\mathbb{R}^+$! We can deform the contour to go slightly above or below the real axis. But an ambiguity remains

$$s_+(F)(g) - s_-(F)(g) = 2\pi \mathrm{i} A g^{-1} \mathrm{e}^{-A/g} \tag{1.5}$$



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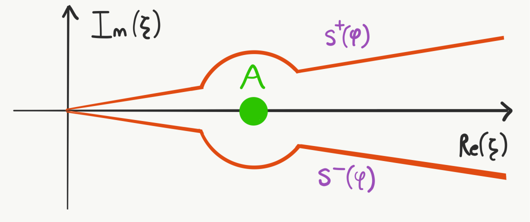

The transseries

Ambiguities can be cancelled by non-perturbative sectors. The “true” function is then given by a **trans-series**

$$\varphi(z) = \sum_{k\geq 0} c_k z^k + \sum_i \sigma_i \mathrm{e}^{-A_i/z} \sum_{k\geq 0} c_k^{(i)} z^k + \cdots \tag{1.6}$$

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In order to cancel ambiguities the **Stokes constants** σ_i must depend themselves on the ray in the complex plane where we perform the Borel summation.

The Borel transform of the perturbative part knows about the transseries sectors! E.g. the position of poles and branch cuts match the weights A_i.

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- Often, large order is enough to specify the structure of the transseries (e.g. the A_i and $c_k^{(i)}$), though not the numerical values of the Stokes constants σ_i (counterexamples: Di Pietro et al. 2021, Kozcaz et al. 2018).

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- Sometimes we are able to construct the transseries from first principles, e.g. the semi-classic expansion of a path integral. But this is not always the case and there is no general framework.

Work to do

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Renormalons in a nutshell

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But a series can also diverge because individual Feynman diagrams through their momenta integration become too big. We call this a **renormalon** effect (discovered in renormalizable theories, and baptised in analogy with instantons.).

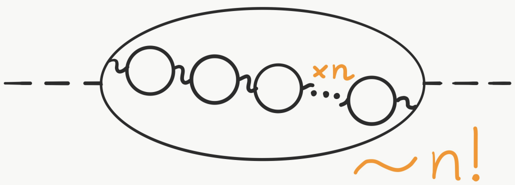

Figure 2: A typical renormalon diagram in particle physics.

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How can we know if there is a renormalon effect in a theory?

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"A man grows stale if he works all the time on insoluble problems, and a trip to the beautiful world of one dimension will refresh his imagination better than a dose of LSD." - Freeman Dyson

The ground state of integrable models can be described by an **Bethe Ansatz integral equation** (BAIE) of the form

$$f(\theta) + \int_{-B}^{B} K(\theta - \theta')f(\theta')\,d\theta' = g_0(\theta). \tag{2.7}$$

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A method by **Volin** (2009) gives $f(\theta)$ as a power series in the weak coupling limit ($B \gg 1$) to arbitrarily high order (40-50 exactly, Abbot et al. get 2000 numerically in a special case.).

Example of the Gaudin-Yang model

Gaudin-Yang model

The Gaudin-Yang model (1967)

- is a **one dimensional** $\mathcal{N}$-particle gas of **spin 1/2 fermions** in a circle of length L. We take the limit of finite $n = \mathcal{N}/L$ and $\mathcal{N}, L \to \infty$.
- has an **attractive** interaction given by a δ-function interaction potential of coupling g. We normalize to the dimensionless coupling $\gamma = g/n$.
- is **integrable**, its ground state is characterised by a BAIE

$$\frac{f(x)}{2} + \frac{1}{2\pi}\int_{-B}^{B} \frac{f(y)\mathrm{d}y}{(x-y)^2+1} = 1, \qquad -B < x < B. \tag{3.8}$$

- can be generalised to N_s spin components.

Volin's method in a flash

The method focus of the resolvent

$$R(x) = \int_{-B}^{B} \frac{f(y)}{x-y} \mathrm{d}y \tag{3.9}$$

from which we obtain f and its moments

$$f(x) = -\frac{1}{2\pi \mathrm{i}}(R(x+\mathrm{i}\epsilon) + R(x-\mathrm{i}\epsilon)), \quad R(x) = \sum_{k\geq 0} \frac{1}{x^{k+1}} \int_{-B}^{B} y^k f(y) \mathrm{d}y\,. \tag{3.10}$$

And the equation can be written in terms of the resolvent and it's shifts $Du(x) = u(x+\mathrm{i})$ as

$$(1+D)R(x+\mathrm{i}\epsilon) - (1+D^{-1})R(x-\mathrm{i}\epsilon) = -4\pi\mathrm{i}, \tag{3.11}$$

Two regimes picture

The equation for the resolvent can be solve in two limits.

The **edge regime** $x = B + z, B \to \infty$ (z finite) reduces to the solution of a Wiener-Hopf equation, but it can only specify the coefficients at leading order in $1/B$.

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The **bulk regime** $x = Bu, B \to \infty$ (u finite) is tackled with an ansatz fixed by the complex properties of the resolvent and the perturbative expansion of the equation, but there is an homogeneous solution which is not fixed.

Comparing the two limits fixes the unknown parts of both!

Asymptotics of the series

The weak coupling expansion of the ground state energy

$$e(\gamma) = \pi^2 \frac{\int_{-B}^{B} x^2 f(x)\mathrm{d}x}{\left(\int_{-B}^{B} x f(x)\mathrm{d}x\right)^3} = \sum_{k\geq 0} c_k \gamma^k$$
$$= \frac{\pi^2}{12} - \frac{\gamma}{2} + \frac{\gamma^2}{6} - \frac{\zeta(3)}{\pi^4}\gamma^3 - \frac{3\zeta(3)}{2\pi^6}\gamma^4$$
$$- \frac{3\zeta(3)}{\pi^8}\gamma^5 - \frac{5(5\zeta(3)+3\zeta(5))}{4\pi^{10}}\gamma^6 - \frac{3\left(12\zeta(3)^2+35\zeta(3)+75\zeta(5)\right)}{8\pi^{12}}\gamma^7 + \mathcal{O}\left(\gamma^8\right) \tag{3.12}$$

Resurgent analysis

Using more than 50 coefficients

$$c_k \sim -\frac{1}{\pi}(\pi^2)^{-k+1}\Gamma(k-1), \qquad k \gg 1. \tag{3.13}$$

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$$c_k \sim -\frac{1}{\pi}(\pi^2)^{-k+1}\Gamma(k-1), \qquad k \gg 1. \tag{3.13}$$

This leads to a Borel singularity at $\zeta = \pi^2$ and a non-perturbative ambiguity

$$-2\gamma \mathrm{e}^{-\pi^2/\gamma} \tag{3.14}$$

This matches the physics of the superconductor behaviour of the system.

Comparing with numerical solution of the integral equation we can specify the first Stokes constant

$$\sigma_1 = \pm \mathrm{i}\,. \tag{3.15}$$

Divergent diagrams

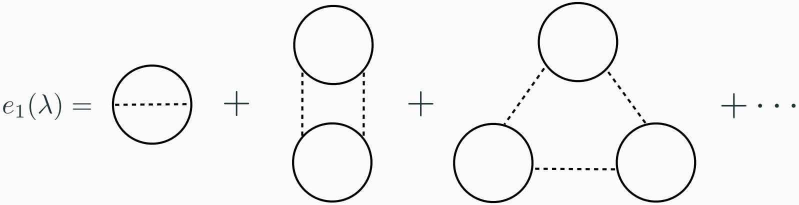

Figure 3: Ring diagrams all the way down.

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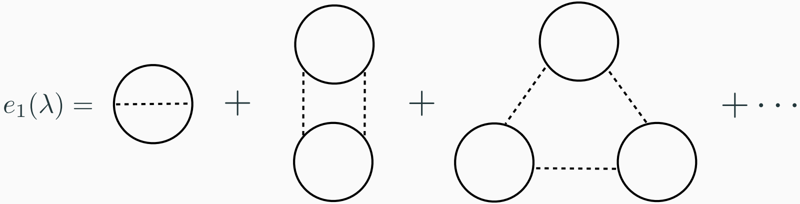

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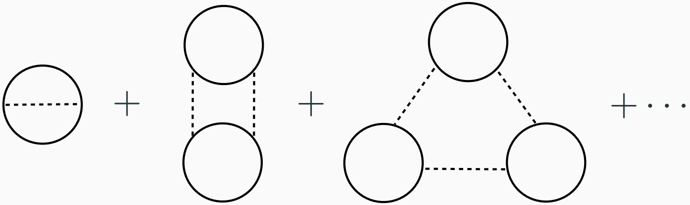
$e_1(\lambda) =$ 

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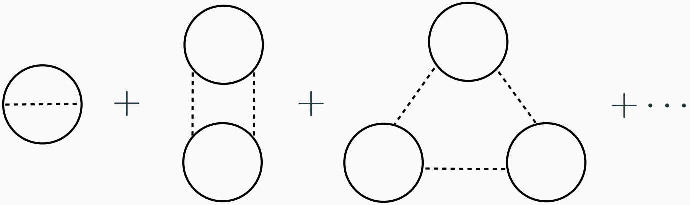

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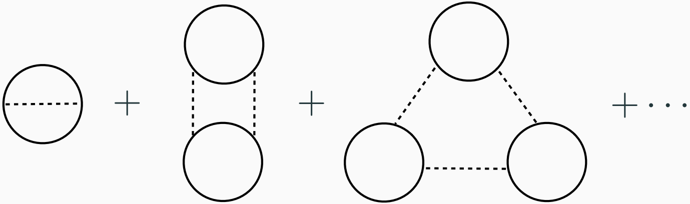

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Superconductivity manifests in the weak coupling expansion as a **renormalon**! We also identified evidence for this in other systems, even non-integrable ones.

Other QFTs

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Can we do better?

Future insights (work in progress)

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However, this method is not as good to calculate large order in perturbative series (for now?).

Conclusion

Take-home ideas

- Resurgence is a useful tool to make sense of both perturbative and non-perturbative physics in quantum field theory.
- Both instantons and renormalons are needed in quantum theory but the latter still need more study.
- Renormalons exist far beyond renormalizable theories.
- Integrability is a powerful tool to study perturbation theory and resurgence analysis.
- Condensed matter is a interesting lab for renormalon effects. Non-perturbative effects are more isolated and possibly experimentally testable.

Thank you!