Resurgence and BPS invariants

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1908.07065: Grassi, Gu, Marino
2007.10190: Garoufalidis, Gu, Marino
2012.00062: Garoufalidis, Gu, Marino
2104.07437: Gu, Marino

How to make sense of an asymptotic series

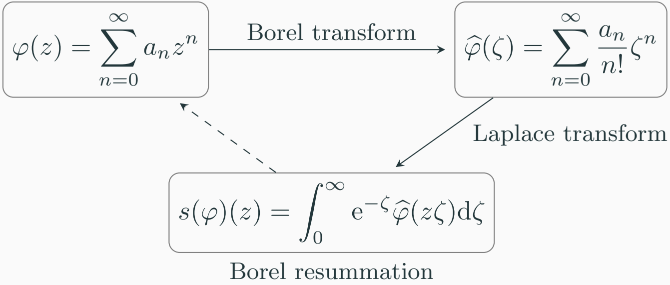
A typical Gevrey-1 asymptotic series in physics

$$\varphi^{(j)}(z) = \sum_{n=0}^{\infty} a_n^{(j)} z^n, \qquad a_n^{(j)} \sim \frac{n!}{A_j^n}.$$

- How do we "sum" the asymptotic series?
- Is it possible to connect the series to the (path) integral and the series from other saddles?

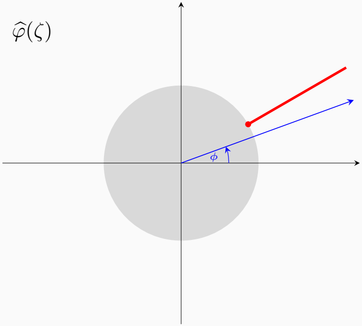
Resurgence theory

Borel resummation



The Borel resummation $s(\varphi)(z)$ reproduces the series $\varphi(z)$ in small z expansion

Borel resummation

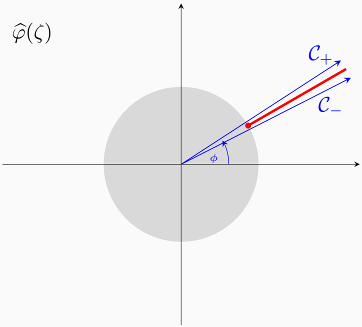


If there is no obstruction along $\phi = \arg z$ in the ζ-plane (Borel plane),

$$s(\varphi)(z) = \int_0^\infty e^{-\zeta}\widehat{\varphi}(e^{i\phi}|z|\zeta)d\zeta,$$

is a well defined integral.

Lateral Borel resummation



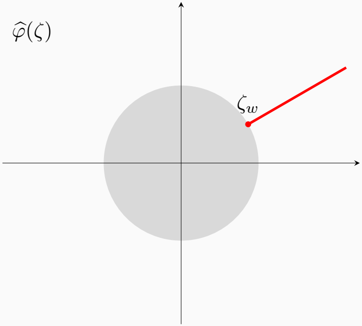
If there is obstruction along $\phi = \arg z$ (Stokes ray), one defines the lateral Borel resummations

$$s_{\pm}(\varphi)(z) = \int_0^{\mathrm{e}^{\mathrm{i}0^{\pm}}\infty} \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)\mathrm{d}\zeta,$$

and Stokes discontinuity

$$\mathrm{disc}(\varphi)(z) = s_+(\varphi)(z) - s_-(\varphi)(z).$$

Resurgent functions



Expansion near ζ_w

$$\widehat{\varphi}(\zeta_w + \xi) = -\mathsf{S}_w \frac{\log(\xi)}{2\pi} \widehat{\varphi}_w(\xi) + \widehat{r}_w(\xi)$$

with regular functions $\widehat{r}_w(\xi)$ and

$$\widehat{\varphi}_w(\xi) = \sum_{n \geq 0} a_{n,w} \xi^n,$$

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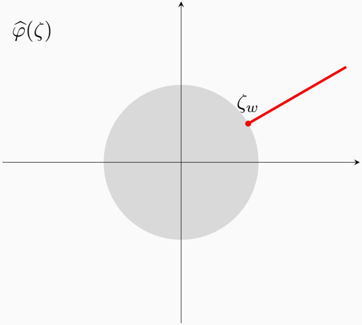

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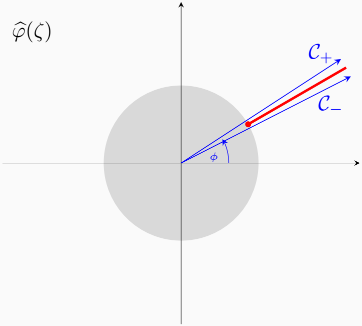
$$\widehat{\varphi}_w(\xi) = \sum_{n \geq 0} a_{n,w} \xi^n,$$

which is regarded as Borel transform of a resurgent series

$$\varphi_w(z) = \sum_{n \geq 0} a_{n,w} z^n, \quad \widehat{a}_{n,w} = \frac{a_{n,w}}{n!}.$$

Resurgent functions and Stokes discontinuity



Resurgence at ζ_w

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implies Stokes discontinuity

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with Stokes constant S_w.

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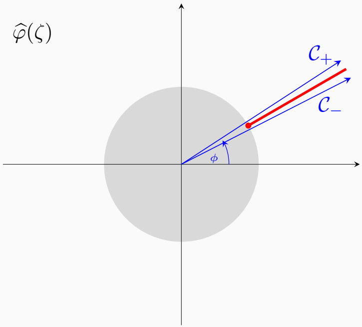

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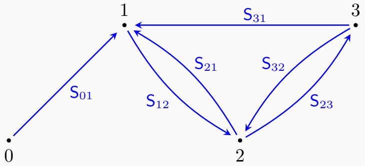
with Stokes constant S_w.

new saddle: $A_w - A_0 = \zeta_w$

Minimal resurgent structure

Starting from one asymptotic series, one finds recursively resurgent asymptotic series, which form a *minimal resurgent structure*:

$$\varphi_0(z) \to \{\varphi_w(z)\} \to \{\mathsf{S}_{ww'}\}$$



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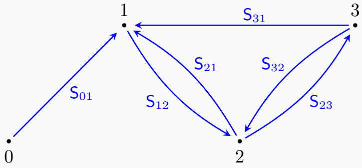
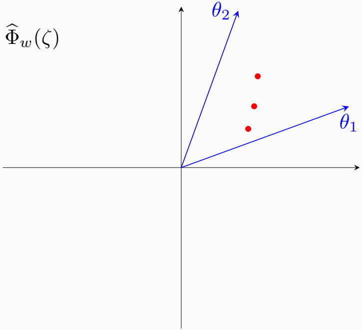


- $\{\mathsf{S}_{ww'}\}$ are new invariants, which are *non-perturbative* in nature.
- Sometimes $\mathsf{S}_{ww'}$ can be interpreted as counting of BPS states.

Stokes automorphism



(Local) Stokes automorphism $\mathfrak{S}_\phi$ at angle ϕ acting on trans-series $\Phi_w(z) = \mathrm{e}^{-A_w/z}\varphi_w(z)$

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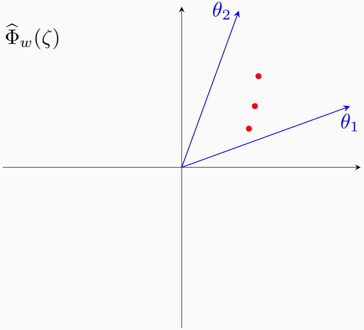

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Global Stokes automorphism between two angles

$$\mathfrak{S}_{\theta_1,\theta_2} = \overleftarrow{\prod_{\theta_1<\phi<\theta_2}} \mathfrak{S}_\phi.$$

- Ordered product;
- Unique factorisation.

Comparison with Wall-Crossing formula

Let us recall the Wall-Crossing formula of Kontsevich-Soibelman for BPS invariants.

- Let Γ be lattice of elec./mag. charges with pairing $\langle , \rangle$, functions $\mathcal{X}_\gamma : \mathcal{M} \to \mathbb{C}^*$.

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- Define symplectomorphism [Kontsevich,Soibelman][Gaiotto,Moore,Neitzke]

$$\mathfrak{S}(\phi) = \prod_{\gamma_{\mathrm{BPS}:\arg(-Z_{\gamma_{\mathrm{BPS}}})=\phi}} \mathcal{K}_{\gamma_{\mathrm{BPS}}}$$

where $\mathcal{K}_{\gamma_{\mathrm{BPS}}}$ acts by

$$\mathcal{K}_{\gamma_{\mathrm{BPS}}} : \mathcal{X}_\gamma \to \mathcal{X}_\gamma (1 - \sigma(\gamma_{\mathrm{BPS}}) \mathcal{X}_{\gamma_{\mathrm{BPS}}})^{\Omega(\gamma_{\mathrm{BPS}}) \langle \gamma, \gamma_{\mathrm{BPS}} \rangle}$$

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- Global symplectomorphism (spectrum generator)

$$\mathfrak{S}(\theta_1, \theta_2) = \overleftarrow{\prod_{\theta_1 < \phi < \theta_2}} \mathfrak{S}(\phi).$$

  - Ordered product;
  - Unique factorisation.

Stokes constants vs BPS invariants

Stokes constants (if integers!)	BPS invariants
Stokes automorphism	KS symplectomorphism

Example 1: Seiberg-Witten theory

Seiberg-Witten theory and its BPS spectrum

4d $\mathcal{N} = 2$ pure $SU(2)$ theory has moduli space identified with family of spectral curves

[Seiberg,Wittne]

$$p^2 + 2\Lambda^2 \cosh x = 2u$$

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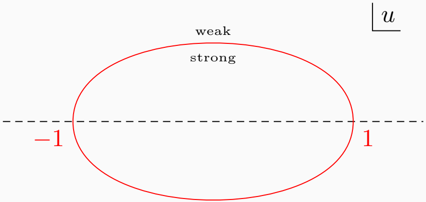
Curve of marginal stability



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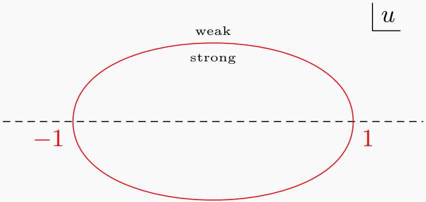


BPS spectrum

- $|u| < 1$: Strong coupling

$$\pm(0,1), \quad \pm(1,1)$$

- $|u| > 1$: Weak coupling

$$\pm(1,0), \quad \pm(\ell,1), \;\; \ell \in \mathbb{Z}$$

Quantum periods

Quantum spectral curve

$$-\hbar^2\psi''(x) + 2\Lambda^2\cosh(x)\psi(x) = E\psi(x)$$

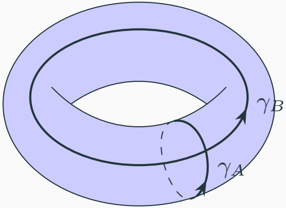
has WKB solutions

$$\psi(x,E) = \exp\left(\frac{\mathsf{i}}{\hbar}\int^x p(x,E;\hbar)\mathrm{d}x\right)$$

Quantum periods

Classical spectral curve

$H_1(\Sigma)$ gives lattice $\Gamma = \mathbb{Z}^2$ with pairing $\langle , \rangle$



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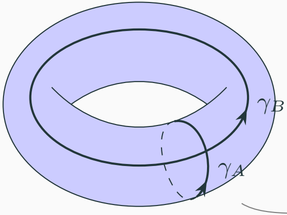

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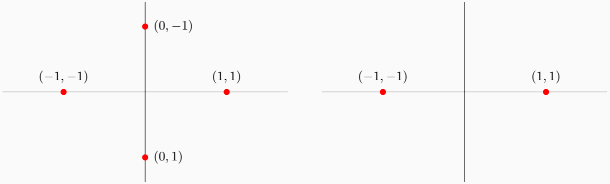
Quantum periods: $\Pi_\gamma(E;\hbar) = \oint_\gamma p(x, E; \hbar)\mathrm{d}x = \sum_{n=0} \Pi_\gamma^{(n)}(E)\hbar^{2n}$

Voros symbols: $\Phi_\gamma(E;\hbar) = \mathrm{e}^{\frac{1}{\hbar}\Pi_\gamma(E;\hbar)} = \mathrm{e}^{\frac{1}{\hbar}\Pi_\gamma^{(0)}(E)} \exp \sum_{n\geq 1} \Pi_\gamma^{(n)}(E)\hbar^{2n-1}$

Stokes automorphism

Borel singularities of quantum periods

- $u = 0$



$\Pi_A(\hbar)$ $\Pi_B(\hbar)$

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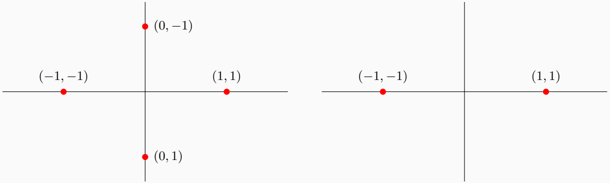

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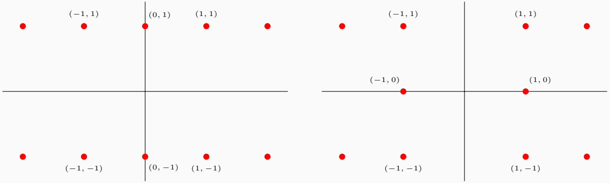
- $u = E/2 = 4$



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Identification

A,B cycles	elec., mag. charges
Saddle points	BPS states
Classical period $\Pi_\gamma^{(0)}$	Central charge Z_γ

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Voros symbol Φ_γ	function $\mathcal{X}_\gamma$
Stokes automorphism	KS symplectomorphism
$\frac{1}{\hbar}\Pi_\gamma \to \frac{1}{\hbar}\Pi_\gamma + \mathsf{S}_{\gamma\gamma'} \log(1 - \sigma_{\gamma'} \mathrm{e}^{\frac{1}{\hbar}\Pi'_\gamma})$	$\mathcal{X}_\gamma \to \mathcal{X}_\gamma \left(1 - \sigma_{\gamma_{\mathrm{BPS}}} \mathcal{X}_{\gamma_{\mathrm{BPS}}}\right)^{\Omega_{\gamma_{\mathrm{BPS}}} \langle \gamma, \gamma_{\mathrm{BPS}} \rangle}$
Stokes constants $\mathsf{S}_{\gamma\gamma'}$	BPS invariants $\Omega_{\gamma_{\mathrm{BPS}}} \langle \gamma, \gamma_{\mathrm{BPS}} \rangle$

Example 2: Complex Chern-Simons theory

Action and saddle points

- Chern-Simons theory with gauge group $SL(2,\mathbb{C})$ and action [Witten][Gukov]

$$S = \frac{t}{8\pi}\int_M \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right) + \frac{\bar{t}}{8\pi}\int_M \mathrm{Tr}\left(\overline{A}\wedge \mathrm{d}\overline{A} + \frac{2}{3}\overline{A}\wedge \overline{A}\wedge \overline{A}\right)$$

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- Saddles are flat connections

$$\mathrm{d}A + A\wedge A = 0, \qquad A\in SL(2,\mathbb{C}),$$

classified via holonomies

$$\rho : H_1(M) \to \mathbb{C}.$$

Non-Abelian saddles and state-integrals

- In complex Chern-Simons non-Abelian flat connections are also important with asymptotic expansion [Dimofte,Gukov,Lenells,Zagier]

$$Z^{(\rho)}(M,\hbar) \sim \exp\left(\frac{1}{\hbar}S_0^{(\rho)} - \frac{1}{2}\delta^{(\rho)}\log\hbar + \sum_{n=0}^{\infty} S_{n+1}^{(\rho)}\hbar^n\right), \quad \hbar = 2\pi/t.$$

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- For hyperbolic 3-manifold M, $\exists$ special non-Abelian flat connection called geometric connection so that (Volume Conjecture)

$$S_0^{(\rho)} = \mathrm{Vol}(M) + \mathsf{i}\,\mathrm{CS}(M)$$

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and $S_{n+1}^{(\rho)}$ $(n \geq 0)$ are in the same algebraic field.
- The asymptotic series $Z^{(\rho)}(M,\hbar)$ for non-Abelian ρ can be computed by state integral [Hikami][Andersen,Kashaev]

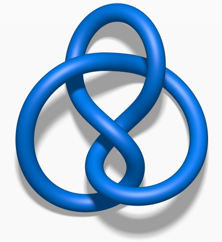
$$Z^{(\rho)}(\hbar) \sim \int_{\mathcal{C}_\rho} P(\Phi_{\mathsf{b}}(v))\mathrm{e}^{\pi\mathsf{i}Q(v)}\mathrm{d}v, \quad \hbar = 2\pi\mathsf{b}^2$$

whose main ingredient is Faddeev's quantum dilogarithm $\Phi_{\mathsf{b}}(v)$.

Example: figure eight complement

- Example: $M = S^3 \backslash \mathbf{4}_1$

$$Z^{(\rho)}(\hbar) \sim \int_{\mathcal{C}_\rho} \Phi_{\mathsf{b}}(v)^2 \mathrm{e}^{-\pi \mathrm{i} v^2} \mathrm{d}v =: I_{\mathcal{C}_\rho}(\mathsf{b}).$$



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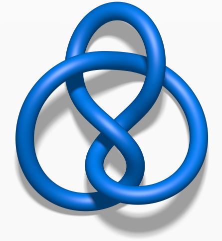

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Non-trivial non-Abelian flat connection

$$\begin{aligned} Z_g(\hbar) =& \mathrm{e}^{\frac{\mathcal{V}}{\hbar}} \left(1 + \frac{11\hbar}{72\sqrt{3}} + \frac{697\hbar^2}{2(72\sqrt{3})^2} + \dots \right), \\ Z_c(\hbar) =& \mathsf{i}\, \mathrm{e}^{-\frac{\mathcal{V}}{\hbar}} \left(1 - \frac{11\hbar}{72\sqrt{3}} + \frac{697\hbar^2}{2(72\sqrt{3})^2} + \dots \right) = \mathsf{i}\, Z_g(-\hbar) \end{aligned}$$

with $\mathcal{V} = \mathrm{Vol}(S^3 \backslash \mathbf{4}_1) = 2\,\mathrm{Im}\,\mathrm{Li}_2(\mathrm{e}^{\pi \mathsf{i}/3})$.

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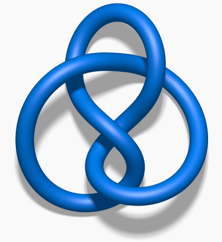

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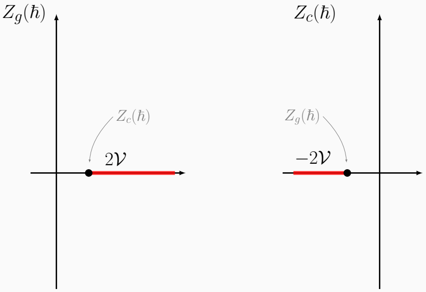
- $I_{\mathbb{R}}(\mathsf{b})$ factorises to holomorphic, anti-holomorphic blocks with $q = \mathrm{e}^{2\pi \mathsf{i} \mathsf{b}^2}$, $\tilde{q} = \mathrm{e}^{-2\pi \mathsf{i} \mathsf{b}^{-2}}$ [Beem,Dimofte,Pasquetti]

$$I_{\mathbb{R}}(\mathsf{b}) \sim G^0(\tilde{q})G^1(q) - \mathsf{b}^{-1} G^1(\tilde{q})G^0(q).$$

Borel singularities

"Classical" Borel singularities [Gukov,Marino,Putrov][Gang-Hatsuda][Garoufalidis-Zagier]

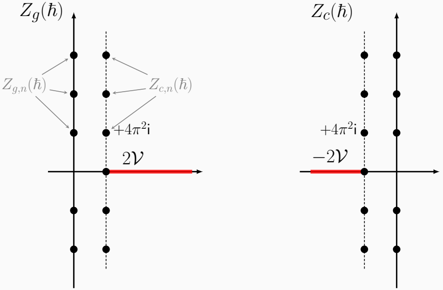


$Z_g(\hbar)$ and $Z_c(\hbar)$ form a minimal resurgent structure.

Borel singularities

More singularities due to multivaluedness of CS action and the state integral potential

[Garoufalidis][Witten][Gukov,Marino,Putrov]



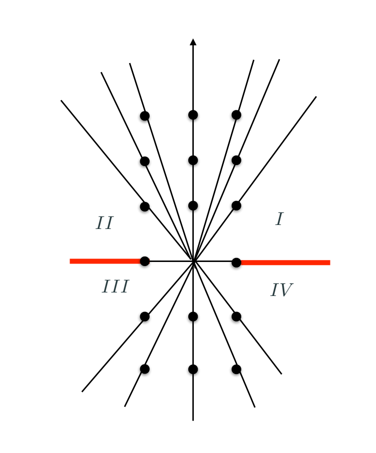
Not one trans-series but a family of trans-series but with the *same* power series

$$Z_{g,n}(\hbar) = Z_g(\hbar)\mathrm{e}^{-n\frac{4\pi^2\mathrm{i}}{\hbar}},$$

$$Z_{c,n}(\hbar) = Z_c(\hbar)\mathrm{e}^{-n\frac{4\pi^2\mathrm{i}}{\hbar}}, \quad n \in \mathbb{Z}$$

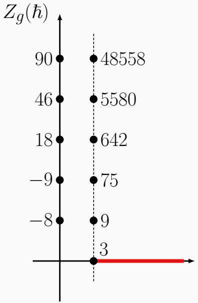
Peacock pattern of Stokes rays

- Stokes rays in the Borel plane for the vector $(Z_g(\hbar), Z_c(\hbar))^T$.



Non-trivial Stokes constants as BPS counting

Despite from trans-series in the same family, the Stokes constants are non-trivial *integers*!



- Generating series of Stokes constants in positive imaginary axis

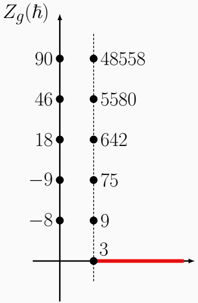
$$\mathsf{S}^{+}_{gg}(q) = 1 - 8q - 9q^2 + 18q^3 + 46q^4 + 90q^5 + \dots, \quad q = \mathrm{e}^{4\pi^2 \mathsf{i}/\hbar}.$$

(Conjecture) It coincides with index $\mathrm{Ind}(0, 1; q)$ of dual 3d superconformal field theory! [Dimofte,Gaiotto,Gukov]

$$\mathrm{Ind}(m, \zeta; q) = \mathrm{Tr}_{\mathcal{H}_m}(-1)^F q^{\frac{R}{2}+j_3}\zeta^e.$$

Non-trivial Stokes constants as BPS counting

Despite from trans-series in the same family, the Stokes constants are non-trivial *integers*!



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- The generating series for the other Stokes constants are also identified with the index with magnetic flux turned on.

Full spectrum of Stokes constants

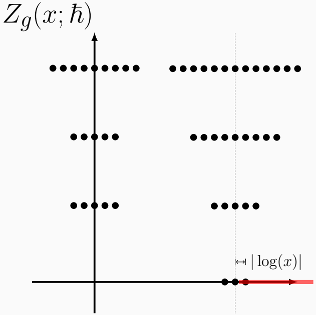
- (Conjecture) Complete set of Stokes constants can be solved!
- The Stokes q-series

$$\mathsf{S}^{\pm}_{\sigma\sigma'}(q) = 1 + \sum_{n=1}^{\infty} \mathsf{S}_{\sigma\sigma';\pm n} q^{\pm n}, \quad \mathsf{S}_{\sigma\sigma';\pm n} \in \mathbb{Z}$$

are given by bilinear expressions in fundamental solutions of the equation

$$y_{m+1}(q) + y_{m-1}(q) - (2 - q^m) y_m(q) = 0$$

Turning on deformation



- Turning on deformation of hyperbolic structure

$$Z_{g,c}(\hbar) \to Z_{g,c}(x;\hbar) \sim \mathrm{e}^{-2\pi \mathrm{i} u^2} \int_{C_\rho} \Phi_{\mathsf{b}}(z)\Phi_{\mathsf{b}}(z+u)\mathrm{e}^{-\pi \mathrm{i}(z^2+4uz)}\mathrm{d}z, \quad x = \mathrm{e}^u.$$

- Generating series of Stokes constants in vertical towers

$$\begin{aligned} \mathsf{S}^{+}_{gg}(q) = & 1 - (2x^{-2} + x^{-1} + 2 + x + 2x^2)q \\ & - (x^{-2} + 2x^{-1} + 3 + 2x + x^2)q^2 + \mathcal{O}(q^3) \end{aligned}$$

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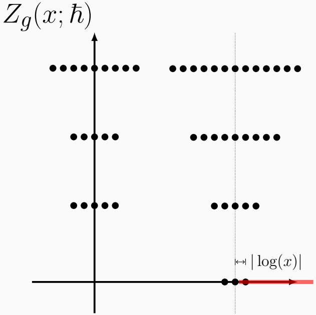

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- They coincide with the index $\mathrm{Ind}(m, x; q)$ with the flavor fugacity turned on.

Full solution of Stokes constants

- The Stokes q-series

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are bilinear expressions in fundamental solutions $y_m(x;q)$ of

$$y_{m+1}(x;q) - (x^2 + x - x^3 q^m) y_m(x;q) + x^3 y_{m-1}(x;q) = 0.$$

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- $y_m(x;q)$ are also solutions to q-difference equation $\widehat{A}(S_x, x, q^m, q)$

$$\widehat{A}(S_x, x, q^m, q) \circ y_m(x;q) = \sum_{j=0}^{2} C_j(x, q^m, q) y_m(q^j x; q) = 0.$$

$\widehat{A}(S_x, x, 1, 1)$ is the A-polynomial with meridiam x^2 and longitude S_x.

Example 3: Topological string theory

Topological string at conifold

Consider topo. string on a (non-)compact Calabi-Yau 3fold X with r Kähler moduli t_i.

- In large volume limit: $t_i \to \infty$

$$F_g(t) = \sum_{\mathbf{d}} N_{g,\mathbf{d}} \mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}, \qquad N_{g,\mathbf{d}} \in \mathbb{Q}$$

  GW invariants $N_{g,\mathbf{d}}$ count numbers of stable maps from worldsheet to X.

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  GW invariants $N_{g,\mathbf{d}}$ count numbers of stable maps from worldsheet to X.

- In maximal conifold point: $t_i \to 0$

$$\mathcal{F}_g(\lambda) = \mathcal{F}_g^{\mathrm{s}}(\lambda) + \mathcal{F}_g^{\mathrm{r}}(\lambda)$$

  where the regular part

$$\mathcal{F}_g^{\mathrm{r}}(\lambda) = \sum_{n_i \geq 0} c_{g;\{n_i\}} \prod_i \lambda_i^{n_i}$$

  The conifold GW invariants $c_{g;\{n_i\}}$, which are in the same algebraic field, have no clear geometric meaning (yet).

Resurgence of topological string?

It is difficult to study the resurgence of total free energy

$$\mathcal{F}(\lambda, g_s) = \sum_{g \geq 0} \mathcal{F}_g(\lambda) g_s^{2g-2}$$

- According to gauge/gravity correspondence, this is the 't Hooft limit of a dual $\prod_i SU(N_i)$ gauge theory

$$\lambda_i = N_i g_s, \quad \text{with } N_i \to \infty, g_s \to 0.$$

- Example: topological string on resolved conifold is dual to $SU(N)$ Chern-Simons theory.

Resurgence at finite N

- We study instead the resurgence of *conifold trans-series* at finite N

$$\Phi_N(g_s) = \exp \mathcal{F}_N^{\mathrm{r}}(g_s) \sim \mathrm{e}^{\frac{1}{g_s}\sum_i N_i \mathcal{V}_i}(1 + \ldots), \quad |g_s| \ll 1,$$

where

$$\mathcal{F}_N^{\mathrm{r}}(g_s) = \sum_{g \geq 0} \mathcal{F}_g^{\mathrm{r}}(N g_s) g_s^{2g-2}.$$

and $\mathcal{V}_i$: Kähler moduli at conifold point.

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- Find the minimal resurgent structure

$$\Phi_{\sigma_1;N}(g_s) := \Phi_N(g_s) \to \{\Phi_{\sigma;N}\} \to \{\mathsf{S}_{\sigma\sigma';N}\}$$

TS/ST correspondence

- Consider models with only one modulus with mirror curve Σ. One obtains a trace class operator ρ_X by quantising Σ [Aganagic,Cheng,Dijkgraaf,Krefl,Vafa][Grassi,Hatsuda,Marino][Kashaev,Marino]

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- The fermionic trace $Z_N(\hbar) \sim \mathrm{Tr}\rho_X^N + \dots$ of ρ_X is related to the series $\Phi_N(g_s)$ [Grassi-Hatsuda-Marino]

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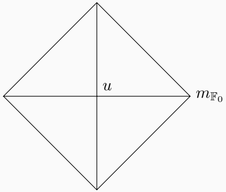
- First similarity with complex Chern-Simons: "state-integral" in terms of quantum dilogarithm

$$Z_N(\hbar) = \int_{\mathbb{R}} P(\Phi_{\mathsf{b}}(x)) \mathrm{e}^{\pi \mathrm{i} Q(x)} (\dots) \mathrm{d}x$$

Example: local $\mathbb{F}_0$

Toric diagram



Mirror curve

$$e^{x} + m_{\mathbb{F}_0} e^{-x} + e^{y} + e^{-y} + \tilde{u} = 0$$

- One true Kahler modulus asso. to comp. 4-cycle $\mathbb{F}_0$
- One mass parameter $m_{\mathbb{F}_0}$ asso. to non-comp. 4-cycle; we set $m_{\mathbb{F}_0} = 1$.

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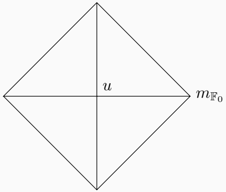

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Conifold trans-series at $N = 1$ [Haghighat,Klemm,Rauch]

$$\Phi_{\sigma_1;1}(g_s) = \mathrm{e}^{\frac{\mathcal{V}_{\mathbb{F}_0}}{g_s}}\left(1 + \tfrac{\pi^2}{24}g_s + \tfrac{73\pi^4}{1152}g_s^2 + \ldots\right), \quad \mathcal{V}_{\mathbb{F}_0} = \frac{2C}{\pi^2}.$$

$C = \operatorname{Im}\operatorname{Li}_2(\mathsf{i})$: Catalan's constant.

State integral

- The trace class operator is

$$\rho_{\mathbb{F}_0} = \mathsf{O}_{\mathbb{F}_0}^{-1}, \quad \mathsf{O}_{\mathbb{F}_0} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}$$

  with commutation relation $[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$.

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- The first trace has integral representation [Kashaev,Marino,Zakany]

$$Z_1(\hbar) = \mathrm{Tr}\rho_{\mathbb{F}_0}(\mathsf{b}) = \frac{1}{2\mathsf{b}} \int_{\mathbb{R}} \frac{\Phi_{\mathsf{b}}(x + \mathsf{i}\mathsf{b}/4)^2}{\Phi_{\mathsf{b}}(x - \mathsf{i}\mathsf{b}/4)^2} \mathrm{e}^{\mathsf{i}\mathsf{b}x} \mathrm{d}x, \quad \hbar = \pi \mathsf{b}^2.$$

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It has two saddle points with fluctuation

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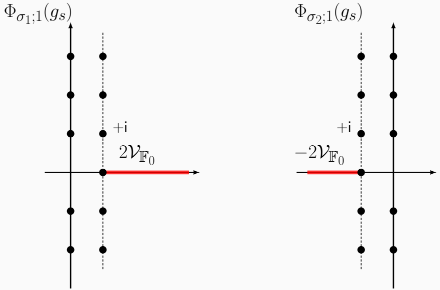
- Second similarity with complex CS: $\mathrm{Tr}\rho_{\mathbb{F}_0}$ factorises to holomorphic and anti-holomorphic blocks

$$\mathsf{Tr}\rho_{\mathbb{F}_0}(\mathsf{b}) = -\frac{\mathsf{i}}{2}\left(G_0(q)g_0(\tilde{q}) + 8\mathsf{b}^{-2} g_0(q)\widetilde{G}(\tilde{q})\right).$$

Resurgent structure

Third similarity: Vertical towers of Borel singularities



Two families of trans-series with the *same* power series

$$\Phi_{\sigma_1,n;1}(g_s) = \Phi_{\sigma_1;1}(g_s)\mathrm{e}^{-n\frac{\mathrm{i}}{g_s}},$$

$$\Phi_{\sigma_2,n;1}(g_s) = \Phi_{\sigma_2;1}(g_s)\mathrm{e}^{-n\frac{\mathrm{i}}{g_s}}.$$

Stokes constants

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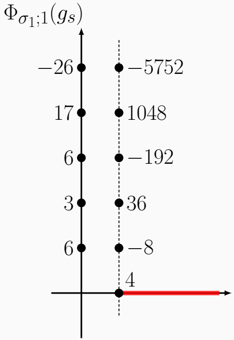

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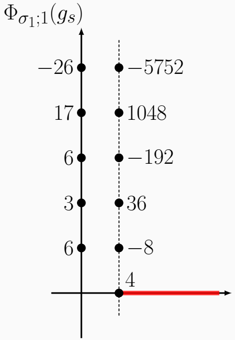

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$$\mathsf{S}^{\pm}_{\sigma\sigma'}(q) = \sum_{n=1}^{\infty} \mathsf{S}_{\sigma\sigma';\pm n} q^{\pm n/2}$$

are given by bilinear combinations in fundamental solutions of the q-difference equation

$$(1-q^{\frac{3}{2}}x)y(q^2x;q) - q^{1/4}(2-qx)y(qx;q) + q^{\frac{1}{2}}(1-q^{\frac{1}{2}}x)y(x;q) = 0$$

evaluated at $x = 1$.

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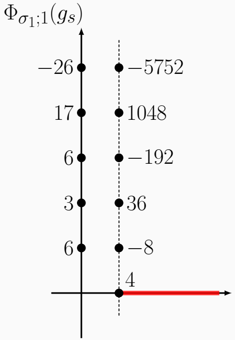

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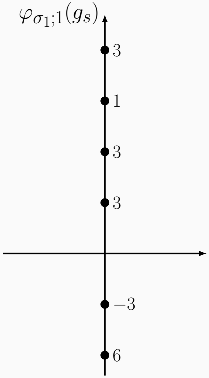
What are the BPS states they are counting?

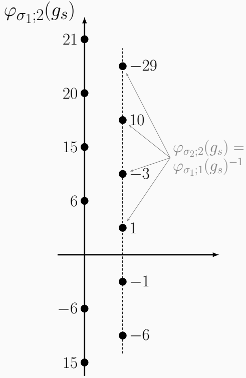
Local $\mathbb{P}^2$

$N = 1$



$\varphi_{\sigma_1;1}(g_s) \sim Z_1$

$N = 2$



$\varphi_{\sigma_1;2}(g_s) \sim Z_2$

$\varphi_{\sigma_2;2}(g_s) = \varphi_{\sigma_1;1}(g_s)^{-1}$

Conclusions and open questions

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- Stokes constants define new *non-perturbative* invariants.
- In some models (SW theory, complex Chern-Simons, conifold topological string) they are non-trivial integers and are BPS countings,
- and they can be solved completely.

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Open questions

- Proof or physical justification of BPS interpretation of Stokes constants in complex Chern-Simons? [3d-3d correspondence]
- Enumerative meaning of the integer Stokes constants in conifold topological string?
- Pattern of Borel singularities as N grows larger in conifold topological string?
- Resurgent structure of conifold topological string on compact Calabi-Yau?
- Solution to Riemann-Hilbert problem related to Stokes automorphism? [Gaiotto,Moore,Neitzke][Bridgeland]

Thank you for your attention!