On resurgent series and their stability under multiplication and Moyal star product

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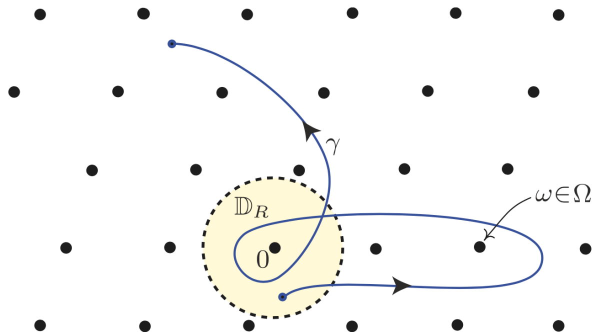
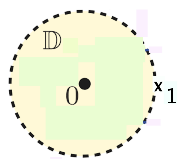
DEFINITION (Ecalle 1981) Let $\Omega \subset \mathbb{C}$ be closed and discrete.
$\widehat{\varphi}(\xi) \in \mathbb{C}\{\xi\}$ is called Ω-continuable if one can follow its analytic continuation along any path starting in its disc of convergence and avoiding Ω.



Elementary examples: Meromorphic functions, algebraic functions.
Here, principal branch assumed to be regular at 0, but maybe $0 \in \Omega$:
$\widehat{\varphi}_1(\xi) := -\frac{1}{\xi}\log(1-\xi)$ and $\mathrm{Li}_2(\xi) := \int_0^\xi \widehat{\varphi}_1(\xi_1)\,\mathrm{d}\xi_1$ are Ω-continuable with $\Omega := \{0,1\}$.



Example related to the Stirling series:

$$\widehat{\mu}(\xi) := \xi^{-2}\Big(\frac{\xi}{2}\coth\frac{\xi}{2} - 1\Big) = \tfrac{1}{12} - \tfrac{1}{360}\tfrac{\xi^2}{2!} + \tfrac{1}{1260}\tfrac{\xi^4}{4!} - \ldots \in \mathbb{C}\{\xi\}$$

meromorphic, poles on $2\pi\mathrm{i}\mathbb{Z}^* \leadsto 2\pi\mathrm{i}\mathbb{Z}^*$-continuable.
Its Laplace transform is the log of the normalized Gamma function

$$\mu(z) = \log\left(\frac{\Gamma(z)}{\sqrt{2\pi}z^{z-\frac{1}{2}}\mathrm{e}^{-z}}\right) \sim \tilde{\mu}(z) = \frac{1}{12}z^{-1} - \frac{1}{360}z^{-3} + \frac{1}{1260}z^{-5} + \ldots$$

A less elementary example: Denote by $W_0(x) > W_{-1}(x)$ the real branches of the Lambert W fcn for $x \in (-\mathrm{e}^{-1}, 0)$ (solving $w\,\mathrm{e}^w = x$).

$$\widehat{\lambda}(\xi) := \frac{1}{\sqrt{2\pi}}(W_0 - W_{-1})(-\mathrm{e}^{-1-\xi}) = \tfrac{\xi^{1/2}}{\Gamma(3/2)} + \tfrac{\xi^{3/2}}{12\Gamma(5/2)} + \tfrac{\xi^{5/2}}{288\Gamma(7/2)} + \ldots \in \xi^{1/2}\mathbb{C}\{\xi\}$$

is $2\pi\mathrm{i}\mathbb{Z}$-continuable *[slight generalization needed]*, its Laplace transform is

$$z^{-3/2}\mathrm{e}^{\mu(z)} = \frac{\Gamma(z)}{\sqrt{2\pi}z^{z+1}\mathrm{e}^{-z}} \sim z^{-3/2}\mathrm{e}^{\tilde{\mu}(z)} = z^{-3/2} + \frac{1}{12}z^{-5/2} + \frac{1}{288}z^{-7/2} + \ldots$$

[One can deal with $\xi^c\mathbb{C}\{\xi\}$ provided $\Re e\, c > -1$.]

DEFINITION (Ecalle 1985, Candelpergher-Nosmas-Pham 1993)

$\widehat{\varphi}(\xi) \in \mathbb{C}\{\xi\}$ *[or $\xi^c\mathbb{C}\{\xi\}$ or sum of such]* is called endlessly continuable if, for every $L > 0$, there exists a finite $\Omega_L \subset \mathbb{C}$ such that one can follow the analytic continuation along any path of length $\leqslant L$ starting in the disc of convergence and avoiding Ω_L.

The singular locus $\Omega := \bigcup_{L>0} \Omega_L$ may be dense in $\mathbb{C}$ but you only see finitely many obstacles at a time. This more general situation occurs in WKB (but if Ω closed discrete, then we're back to Ω-continuability).

DEFINITION A resurgent series is any formal series whose formal Borel transform is an endlessly continuable germ.

A resurgent function is any function which can be obtained from an endlessly continuable germ by Laplace transform (not necessarily in the direction of $R_{>0}$).

$$\begin{tikzcd} \text{Resurgent function } \varphi(z) & \\ & \hat{\varphi}(\xi) \text{ endlessly continuable} \arrow[ul, "\text{Laplace } \mathscr{L}^{\theta}"'] \\ \text{Resurgent series } \widetilde{\varphi}(z) \arrow[uu, dashed] \arrow[ur, "\text{Borel } \mathscr{B}"] & \end{tikzcd}$$

THEOREM (Ecalle) Resurgent series are stable under multiplication and nonlinear operations, e.g. substitution into a convergent power series.

So it's not a surprise that $\mathrm{e}^{\tilde{\mu}(z)}$, the exponential of the Stirling series, is also resurgent. This also "explains" why resurgent series are so abundant in nature...

A related topic is "alien calculus": the use of Ecalle's alien operators, which behave nicely w.r.t. multiplication and allow to measure singularities in the Borel plane so as to handle the Stokes phenomena...

The proof of the theorem requires the analysis of the counterpart of multiplication in the Borel plane, which is the *convolution product*.

Resurgent series are stable under multiplic$^\circ$ $\Leftrightarrow$ endlessly continuable fcns are stable under convolution.

We now review the basic arguments behind the theorem: *How to follow the analytic continuation of the convol$^\circ$ of endlessly continuable fcns?*

Similar arguments allow to handle the Hadamard product of endlessly continuable fcns = resurgent version of the Hadamard mult$^\circ$ thm [1898].

This will serve as a preparation for the study of the Moyal star product of resurgent series in the last part of the talk.

DEFINITION The convolution product of two germs is

$$f * g(\xi) := \mathscr{B}\big((\mathscr{B}^{-1}f)(\mathscr{B}^{-1}g)\big) = \int_0^\xi f(\xi_1)g(\xi-\xi_1)\,\mathrm{d}\xi_1.$$

The Hadamard product of $f(\xi) = \sum_{n\geqslant 0} a_n\xi^n$ and $g(\xi) = \sum_{n\geqslant 0} b_n\xi^n$ is

$$f \odot g(\xi) := \sum_{n\geqslant 0} a_n b_n \xi^n = \oint_{C_\rho} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} f(\zeta) g\big(\tfrac{\xi}{\zeta}\big),$$

where C_ρ = anticlockwise circle of radius ρ, $0 < \rho < R_f$, $|\xi| < \rho R_g$.

THEOREM Given $A, B \subset \mathbb{C}$ closed and discrete, f A-continuable, g B-continuable,

(i) $\Omega_1 := \{0\} \cup (A\cdot B)$ is closed and discrete, $f \odot g$ is Ω_1-continuable;

(ii) if $\Omega_2 := A \cup B \cup (A+B)$ is closed and discrete, then $f * g$ is Ω_2-continuable.

Part (i) [LSS 2020a] can be viewed as a refinement of the statement *about the principal branch* given by Hadamard and Borel in 1898. Borel also says something about the analytic continuation to other sheets...

The necessity of including 0 among the possibly singular points of $f \odot g$ on the other sheets was noted by Borel (who gives credit to E. Lindelöf for that point). Simple example: $f(\xi) := -\log(1-\xi)$ is $\{1\}$-continuable, but $f \odot f$ is not; in fact, $f \odot f = \mathrm{Li}_2$ is $\{0, 1\}$-continuable.

Basic idea translated from Borel's own words: "this expression of $f \odot g$ stays valid *if one deforms the integration contour without letting it cross any singular point of the integrand*" and, "the contour having been fixed in an arbitrary manner, one obtains the analytic continuation of $f \odot g$ *by moving ξ in the plane, provided the singular points of the integrand do not cross the integration contour*".

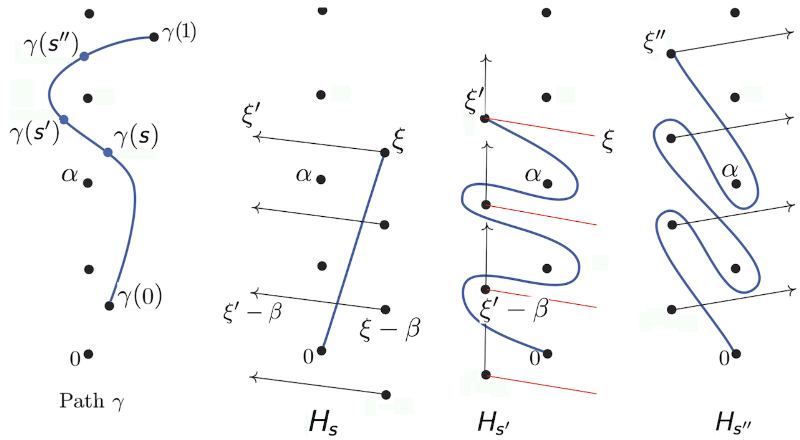
Throughout his 1898 paper, Borel seems to keep in mind the possibility of going to the non-principal sheets and dealing with multivalued analytic continuation, yet reluctantly so, since when he explicitly mentions that possibility he tends to recommend to discard it! ("It seems to us useless to insist on the latter point... one would be led to complicated statements...").

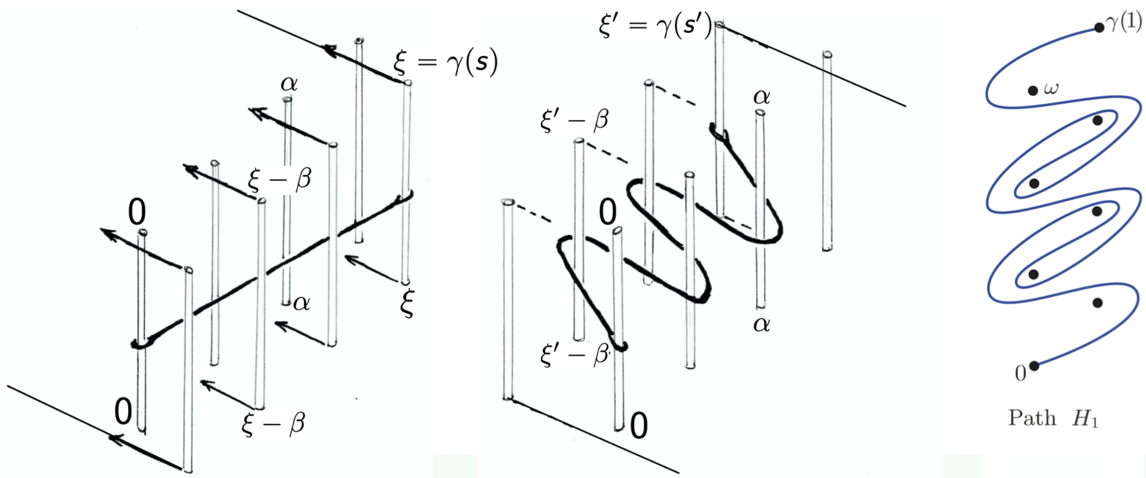
However, he puts in a footnote an idea that has been successfully adapted to study the analytic continuation of the convolution $f * g$ of endlessly continuable germs in [Ecalle 1981], [CNP 1993] (Part (ii)).

Borel writes $f \odot g(\xi) = \oint_C \frac{\mathrm{d}x}{2\pi \mathrm{i} x} f\left(\frac{1}{x}\right) g(\xi x)$, with $C = C_{\rho^{-1}}$ and:

> *"Let us conceive the closed contour C as a flexible extensible thread, the singular points of $f\left(\frac{1}{x}\right)$ as pins stuck into the plane, the singular points of $g(\xi x)$ as pins that travel as ξ moves. It is necessary and sufficient that the thread always part the two systems of pins. Now, this will always be possible, by means of a suitable deformation, if, while travelling, the second pins never come to hit the first ones (...); the thread may acquire a very complicated form, but this is harmless."*

In the case of the convolution product $f * g(\xi) = \int_0^\xi f(\xi_1) g(\xi - \xi_1)\, \mathrm{d}\xi_1$, instead of the closed contour $C = C_{\rho^{-1}}$, it is the line segment $[0, \xi]$ that must be deformed: the analytic continuation of $f * g$ along γ at $\xi = \gamma(s)$ is $\int_{H_s} f(\xi_1) g(\gamma(s) - \xi_1)\, \mathrm{d}\xi_1$ with suitable H_s going from 0 to $\gamma(s)$.





[S 2013,16] for (ii): Contruct explicitly the deformation when $\xi = \gamma(s)$ moves in the complex plane without meeting $A \cup B \cup (A+B)$, by applying to the initial integration contour $[0, \gamma(0)]$ a homeomorphism Ψ_s obtained as the flow at time s of an explicit non-autonomous vector field.

A benefit of such a detailed rigorous proof with respect to the arguments given in [Ecalle 1981] or [CNP 1993] is that it allows for quantitative estimates which, in turn, can be adapted to prove Ecalle's theorem on the stability of the space of resurgent series under nonlinear operations and not only multiplication of two factors [S 2015], [Kamimoto-S 2020].

Adaptation to the case (i) of the Hadamard product [LSS 2020a]: Given $\gamma\colon [0,1] \to \mathbb{C}\backslash\Omega_1$, we wish to construct a continuous family $(\Psi_s)_{s\in[0,1]}$ of Lipschitz homeomorphisms so that $\Psi_0 = \mathrm{Id}_{\mathbb{C}}$ and, for all $s \in [0,1]$,

$$\alpha \in \{0\} \cup A \;\Rightarrow\; \Psi_s(\alpha) = \alpha, \quad \beta \in B\backslash\{0\} \;\Rightarrow\; \Psi_s\Big(\frac{\gamma(0)}{\beta}\Big) = \frac{\gamma(s)}{\beta}. \qquad (1)$$

This way, for ξ near $\gamma(s)$, $\mathrm{cont}_{\gamma|s}(f \odot g)(\xi) = \displaystyle\oint_{\Psi_s(C_\rho)} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} f(\zeta) g\Big(\frac{\xi}{\zeta}\Big).$

Indeed, $\zeta_0 \in C_\rho \;\Rightarrow\; \zeta_0 \notin \{0\} \cup A \;\Rightarrow\; \Psi_s(\zeta_0) \notin \{0\} \cup A$,
and $\zeta_0 \in C_\rho \;\Rightarrow\; \frac{\gamma(0)}{\zeta_0} \notin B \;\Rightarrow\; \frac{\gamma(s)}{\Psi_s(\zeta_0)} \notin B$ (because $\frac{\gamma(s)}{\Psi_s(\zeta_0)} = \beta$ would necessitate $\Psi_s(\zeta_0) = \frac{\gamma(s)}{\beta}$, whence $\zeta_0 = \frac{\gamma(0)}{\beta}$),
therefore $\Psi_s(C_\rho)$ avoids the fixed singular points 0 or α and the moving singular points $\frac{\gamma(s)}{\beta}$.

We may define Ψ_s as the flow map between time $t=0$ and time $t=s$ for the non-autonomous vector field

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = X(t,\zeta) := \eta(\zeta)\tfrac{\zeta}{\gamma(t)}\gamma'(t), \tag{2}$$

with any Lipschitz function $\eta\colon \mathbb{C} \to [0,1]$ such that

$$\alpha \in A\backslash\{0\} \;\Rightarrow\; \eta(\alpha) = 0, \qquad \beta \in B\backslash\{0\} \;\Rightarrow\; \eta\big(\tfrac{\gamma(t)}{\beta}\big) = 1$$

(necessarily $\inf_{t,\beta} \operatorname{dist}\big(\frac{\gamma(t)}{\beta}, A\backslash\{0\}\big) > 0$). The conditions (1) are fulfilled:

$$\alpha \in \{0\} \cup A \;\Rightarrow\; X(t,\alpha) = 0, \qquad \beta \in B\backslash\{0\} \;\Rightarrow\; X\big(t, \tfrac{\gamma(t)}{\beta}\big) = \tfrac{\gamma'(t)}{\beta}.$$

Details for Part (ii) when A and B contain 0 [S 2013,16]:
Given $\gamma : [0,1] \to \mathbb{C}\backslash\Omega_2$ with $\Omega_2 = A + B$, use

$$X(\zeta,t) := \frac{\eta_A(\zeta)}{\eta_A(\zeta) + \eta_B\big(\gamma(t) - \zeta\big)}\gamma'(t)$$

and its flow, where $\eta_A, \eta_B\colon \mathbb{C} \to [0,1]$ have been selected so that

$$\eta_A(\zeta) = 0 \;\Leftrightarrow\; \zeta \in A, \qquad \eta_B(\zeta) = 0 \;\Leftrightarrow\; \zeta \in B.$$

Notice that the denominator stays > 0.

The study of the Hadamard product in [LSS 2020a] was a prelude to...

MOYAL STAR PRODUCT & ALGEBRO-RESURGENT SERIES

Snapshot: *The Borel transform w.r.t. $t = \mathrm{i}\hbar$ of a Moyal star product can be written in terms of the Borel transforms of its factors, $f(\xi, q, p)$ and $g(\xi, q, p)$, and the formula appears as a mixture of convolution with respect to ξ and Hadamard product: it involves the Hadamard product $f(\xi_1, q, p+\xi_3) \odot g(\xi_2, q+\xi_3, p)$ with respect to ξ_3 for fixed q, p, ξ_1, ξ_2, and then a convolution-like integration with respect to ξ_1, ξ_2, ξ_3.*

To handle analytic continuation in such a many-variable context, we need to put restrictions on the singular locus of f and g.

"Algebro-resurgence" [Garay-de Goursac-van Straten 2014]:

$f, g \in \mathbb{C}\{\xi, q, p\}$ have analytic continuation away from a proper algebraic subvariety of $\mathbb{C}^3$ (or $\mathbb{C}^{2N+1}$ when dealing with deformation quantization with N degrees of freedom).

Given a Poisson structure with constant coefficients on $M \subset \mathbb{R}^d$

$$\pi = \sum_{1\leqslant i<j\leqslant d} \pi^{i,j}\partial_i \wedge \partial_j, \qquad \pi^{j,i} = -\pi^{i,j} \in \mathbb{R}, \qquad \partial_i := \tfrac{\partial}{\partial x_i},$$

we can write the Poisson bracket in $C^\infty(M)$ as

$$\{f,g\} = \mu\circ P(f\otimes g), \qquad P := \sum_{1\leqslant i,j\leqslant d} \pi^{i,j}\partial_i\otimes\partial_j, \qquad \mu := \text{multiplication}$$

and the corresponding Moyal star product is defined $C^\infty(M)[[t]]$ by

$$\tilde f \star_M \tilde g = \mu\circ\exp\left(\frac{tP}{2}\right)(\tilde f\otimes\tilde g) = \tilde f\tilde g + \frac{t}{2}\sum_{i,j}\pi^{i,j}\,\partial_i(\tilde f)\,\partial_j(\tilde g)$$
$$+\frac{1}{2!}\left(\frac{t}{2}\right)^2\sum_{i,j,k,\ell}\pi^{i,j}\pi^{k,\ell}\,\partial_i\partial_k(\tilde f)\,\partial_j\partial_\ell(\tilde g) + \cdots$$

It is an associative non-commutative deformation of the product of $C^\infty(M)$ in the direction of π: $f\star_M g - g\star_M f = t\{f,g\} + O(t^2)$.

By a linear change of variables, we reduce the situation to the case of the standard Poisson structure in $M' \subset \mathbb{R}^{2N}$ (where $2N = \text{rank of } [\pi^{i,j}]$), with coordinates $(q_1, \dots, q_N, p_1, \dots, p_N)$.

For instance, if $N = 1$, $\{f, g\} = \partial_p f \, \partial_q g - \partial_q f \, \partial_p g$ and

$$\tilde{f} \star_M \tilde{g} = \tilde{f}\,\tilde{g} + \sum_{k \geqslant 1} \frac{1}{k!} \Big(\frac{t}{2}\Big)^k \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \Big(\partial_p^n \partial_q^{k-n} \tilde{f}\Big) \Big(\partial_p^{k-n} \partial_q^n \tilde{g}\Big).$$

Example: With $f = (1-p)^{-1}$ and $g(q,p)$ analytic both independent of t,

$$(1-p)^{-1} \star_M g(q,p) = \sum_{k \geqslant 0} \Big(\frac{t}{2}\Big)^k (1-p)^{-k-1} \partial_q^k g(q,p)$$

factorially divergent in general, will be resurgent in t under appropriate assumptions on g.

Using slightly modified Borel transform $\mathcal{B}_\star$: $t^k \mapsto \xi^k/k!$ (no shift of exponent), for fixed (q,p),

$$\mathcal{B}_\star\big((1-p)^{-1} \star_M g(q,p)\big) = \sum_{k \geqslant 0} \Big(\frac{\xi}{2}\Big)^k (1-p)^{-k-1} \frac{1}{k!} \partial_q^k g(q,p)$$
$$= \frac{1}{1-p} g\Big(q + \frac{\xi}{2(1-p)}, p\Big)$$

has a finite radius of convergence in ξ (unless $q \mapsto g(q,p)$ extends to $\mathbb{C}$)
is endlessly continuable in ξ if g is endlessly continuable in its 1st arg:
$(1-p)^{-1} \star_M g(q,p)$ is then resurgent in t.

Other example:

$$\mathcal{B}_\star\big(\log(1-p) \star_M \log(1-q)\big) = \log(1-p)\log(1-q) + \mathrm{Li}_2\Big(\frac{\xi}{2(1-q)(1-p)}\Big).$$

DEFINITION (Garay-de Goursac-van Straten) Given $r \geqslant 1$, the set of algebro-resurgent germs in r variables is

$$\widehat{\mathscr{Q}}_r^{\mathcal{A}} := \Big\{ \widehat{\varphi} \in \mathbb{C}\{z_1, \dots, z_r\} \mid \exists V \text{ proper algebraic subvariety of } \mathbb{C}^r \text{ s.t.}$$
$$\widehat{\varphi} \text{ has analytic continuation along any path } \gamma \subset \mathbb{C}^r \backslash V$$
$$\text{having its initial point } \gamma(0) \text{ close enough to } 0 \Big\}.$$

Correspondingly, an algebro-resurgent series is any formal series $\tilde{\varphi}(t, z_2, \dots, z_r) \in \mathbb{C}\{z_2, \dots, z_r\}[[t]]$ s.t. $\mathcal{B}_\star(\tilde{\varphi}) = \widehat{\varphi}(\xi, z_2, \dots, z_r) \in \widehat{\mathscr{Q}}_r^{\mathcal{A}}$.

THEOREM [LSS 2020b] For any constant coefficient Poisson structure in d variables, the Moyal star product of two algebro-resurgent series in $d+1$ variables is an algebro-resurgent series in $d+1$ variables.

For the proof, it is sufficient to consider the standard Poisson structure with $d = 2N$, using canonical coordinates $q_1, \dots, q_N, p_1, \dots, p_N$.

Moreover, it is easier to work with the standard star product

$$\tilde{f} \star_S \tilde{g} := \sum_{k_1, \dots, k_N \geqslant 0} \frac{t^{k_1 + \dots + k_N}}{k_1! \cdots k_N!} \Big(\partial_{p_1}^{k_1} \cdots \partial_{p_N}^{k_N} \tilde{f} \Big) \Big(\partial_{q_1}^{k_1} \cdots \partial_{q_N}^{k_N} \tilde{g} \Big).$$

This turns out to be sufficient because $T\big(\tilde{f} \star_S \tilde{g}\big) = \big(T\tilde{f}\big) \star_M \big(T\tilde{g}\big)$ with a "transition operator"

$$T := \exp\Big(-\frac{t}{2} \sum \partial_{q_j} \partial_{p_j} \Big), \quad T^{-1} = \exp\Big(\frac{t}{2} \sum \partial_{q_j} \partial_{p_j} \Big)$$

and

THEOREM [LSS 2020b]

(i) Algebro-resurgent series in $2N+1$ variables are stable under standard star product.

(ii) Algebro-resurgent series in $2N+1$ variables are stable under the transition operators T and T^{-1}.

The proof relies on integral formulas for the Borel counterparts of the standard star product and the transition operator,

$$f \circledast g := \mathcal{B}_\star\big((\mathcal{B}_\star^{-1} f) \star_S (\mathcal{B}_\star^{-1} g) \big), \qquad \widehat{T}(f) := \mathcal{B}_\star \circ T \circ \mathcal{B}_\star^{-1}(f)$$

LEMMA (N=1 for simplicity) Suppose $f = f(\xi, q, p)$, $g = g(\xi, q, p)$ holomorphic in $\mathbb{D}_\tau \times \mathbb{D}_\tau \times \mathbb{D}_\tau$. Pick $\varepsilon \in (0, \tau)$ and $\varepsilon' \in (0, \varepsilon^2)$.

Then, for any $(\xi, q, p) \in \mathbb{D}_{\varepsilon'} \times \mathbb{D}_{\tau-\varepsilon} \times \mathbb{D}_{\tau-\varepsilon}$ and $\rho \in (\frac{\varepsilon'}{\varepsilon}, \varepsilon)$,

$$f \circledast g(\xi, q, p) = \frac{d^3}{d\xi^3} \int_0^\xi d\xi_1 \int_0^{\xi-\xi_1} d\xi_2 \int_0^{\xi-\xi_1-\xi_2} d\xi_3 \oint_{C_\rho} \frac{d\zeta}{2\pi i \zeta} f(\xi_1, q, p + \frac{\xi_3}{\zeta}) g(\xi_2, q+\zeta, p),$$

$$\widehat{T}^{\pm 1} f(\xi, q, p) = \frac{d}{d\xi} \int_0^\xi d\xi_1 \oint_{C_\rho} \frac{d\zeta}{2\pi i \zeta} f(\xi - \xi_1, q+\zeta, p \mp \frac{\xi_1}{2\zeta}).$$

Remark: The statement of Part (i) can be found in [GGS 2014], however there is a gap in their proof, due to a mistake in their formula for $\circledast$.

These formulas and their multidimensional analogues allow to prove

$$f, g \in \widehat{\mathcal{Q}}^{\mathcal{A}}_{2N+1} \implies f \circledast g \in \widehat{\mathcal{Q}}^{\mathcal{A}}_{2N+1} \text{ and } \widehat{T}^{\pm 1} f \in \widehat{\mathcal{Q}}^{\mathcal{A}}_{2N+1},$$

whence the theorem follows.

Technical lemma

(i) If $f(\xi, q, p), g(\xi, q, p) \in \widehat{\mathcal{Q}}^{\mathcal{A}}_3$, then the formula

$$F(\xi_1, \xi_2, \xi_3, q, p) := \oint_{C_\rho} \frac{d\zeta}{2\pi i \zeta} f(\xi_1, q, p + \frac{\xi_3}{\zeta}) g(\xi_2, q+\zeta, p)$$

defines a germ $F \in \widehat{\mathcal{Q}}^{\mathcal{A}}_5$.

(ii) If P is a polynomial in r variables vanishing at $(0, \dots, 0)$ and $F(z_1, \dots, z_r) \in \widehat{\mathcal{Q}}^{\mathcal{A}}_r$, then the formula

$$G(z, z_2, \dots, z_r) := \int_0^{P(z, z_2, \dots, z_r)} F(z_1, z_2, \dots, z_r) dz_1$$

defines a germ $G \in \widehat{\mathcal{Q}}^{\mathcal{A}}_r$.

(iii) If P is a polynomial in $r-1$ variables vanishing at $(0, \dots, 0)$ and $F(z_1, \dots, z_r) \in \widehat{\mathcal{Q}}^{\mathcal{A}}_r$, then the formula

$$H(z_2, \dots, z_r) := \int_0^{P(z_2, \dots, z_r)} F(z_1, z_2, \dots, z_r) dz_1$$

defines a germ $H \in \widehat{\mathcal{Q}}^{\mathcal{A}}_{r-1}$.

PROSPECTS

(1) Hadamard product: What about the nature of the singularities? The branches of the analytic continuation of $f \odot g$ cannot be singular outside $\{0\} \cup (A \cdot B)$. Cf. case of convolution: the singularities located in $A \cup B \cup (A + B)$ can be analyzed by means of Ecalle's alien operators.

For singular points adherent to the principal sheet: R. Pérez-Marco 2020, "Monodromies of singularities of the Hadamard and eñe product" arXiv:2009.14099 "Local monodromy formula of Hadamard products" arXiv:2011.10497.

(2) What about the singularities of $\mathcal{B}_\star\big(\tilde f \star_M \tilde g\big)$, $\mathcal{B}_\star\big(\tilde f \star_S \tilde g\big)$, $\mathcal{B}_\star \circ T^{\pm 1}\big(\tilde f\big)$?

(3) Can one design a framework for Resurgence in Deformation Quantization that would be more general than Algebro-Resurgence, so as to allow for larger singular loci in the Borel plane?

(4) What about non-constant Poisson structures? Take $\pi^{i,j} = \pi^{i,j}(x_1, \ldots, x_d)$ analytic in $M \subset \mathbb{R}^d$ and use the explicit B.Fedosov's star product (1985) in the symplectic case or M.Kontsevich's star product (1997) in the general case. Might Ecalle's Mould Calculus play a role? (Algebraic combinatorics methods, adapted e.g. for BCH.)

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