BV-BFV quantization & Chern-Simons theory

Motivation: Study of path integral in QFT

Classical field theory: $M \mapsto (F_M, S_M)$

M: spacetime; F_M: space of fields; S_M: action functional [local]

classical physics described by $\delta S_M = 0$

quantum field theory: $O: F_M \to \mathbb{R}$ observable

- partition function $Z \text{ "=" } \int_{F_M} e^{\frac{i}{\hbar} S_M[\phi]} D\phi$
- expectation value $\langle O \rangle \text{ "=" } \frac{1}{Z} \int_{F_M} e^{\frac{i}{\hbar} S_M[\phi]} O(\phi) D\phi$ [Feynman]

e.g. $\begin{cases} F_M = \Omega^1(M) \oplus \Omega^{d-2}(M) \ni (A, B) \\ S_M = \int_M B \wedge dA \end{cases}$

"abelian BF theory"

topological - does not depend on spacetime metric. Focus on this case today.

Usually F_M doesn't admit measures with desirable properties.

$\rightsquigarrow$ Mathematical definition of Z?

i) Atiyah-Segal: Z should behave like a (symmetric monoidal) functor Z_{F_M} : Cob $\to$ Vect

[Fubini for $\int$ = composition

(symmetric monoidal) functor $Z_{Fun} : Cob \to Vect$

[Fubini for $\int$ = composition]

ii) "perturbative": define Z by formally using $\hbar \to 0$ asymptotics

+ formal Gaussian integrals

$\rightsquigarrow Z_{pert}$, formal power series in $\hbar$, compute by Feynman graphs & rules

Natural question:

$$\begin{tikzcd} & (F_M, S_M) \arrow[dl, mapsto] \arrow[dr, dotted, "?"] & \\ Z_{pert} & & Z_{Fun} \end{tikzcd}$$

should depend on $\hbar$

Q1: $\boxed{Z_{Fun} \underset{\hbar \to 0}{\sim} Z_{pert} \ ?}$

Q2: $\boxed{Z_{pert} \underset{\text{Resurgence}}{\rightsquigarrow} Z_{Fun}}$

<u>Ex</u> Chern-Simons theory $\dim M = 3$

$F_M = Conn(M, P)$ P principal G-bundle, $\pi_0(G) = 0, \pi_1(G) = \{e\}$ $\Rightarrow$ trivializable

$$S_M[A] = \int \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A, A]\rangle$$

$\langle\ ,\ \rangle$ invariant non-deg on $\mathfrak{g}$

$$Z \text{ "="} \int_{F_M} e^{\frac{i}{\hbar} S_M} \qquad \boxed{\hbar = \frac{2\pi}{k}} \quad k \in \mathbb{Z}$$

Z_{Fun} : Reshetikhin-Turaev TQFT (conjecturally)

Q1 $\approx$ Asymptotic expansion conjecture.

Need to be able to define Z_{pert} for degenerate theories on spacetimes with boundary $\longrightarrow$ BV-BFV formalism.

2. BV-BFV formalism

To compute Z_{pert}, require nondegenerate critical points of S_M.

E.g. abelian BF theory, CS theory have degenerate critical points.

BV formalism (Batalin, Vilkovisky 1981, 1983)

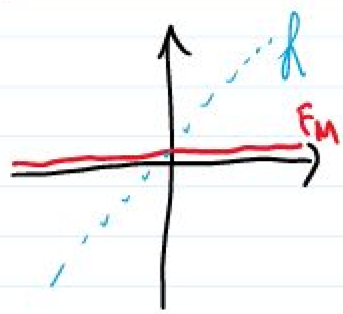
For closed manifolds, have BV formalism: (assume F_M vector space from now on)

B-V give us
- $\mathcal{F}_M$ $\mathbb{Z}$-graded vector space $F_M \subset (\mathcal{F}_M)_0$
- $\boxed{\mathcal{S}_M} \in \mathcal{O}(\mathcal{F}_M)$ $\quad \mathcal{S}_M\big|_{F_M} = S_M$
- ω_M symplectic structure, $\deg = -1$

s.t. $(\mathcal{S}_M, \mathcal{S}_M)_\omega = 0$ [classical Master Equation]

$\Rightarrow Q = (S, \cdot)$ satisfies $\boxed{Q^2 = 0}$

Suppose we have Lagrangian $\boxed{\mathcal{L} \subset \mathcal{F}_M}$ s.t. $\mathcal{S}|_{\mathcal{L}}$ has non-degenerate critical points. Then, define



"BV integral"

$$\boxed{Z = \int_{\mathcal{L}} e^{i/\hbar \, \mathcal{S}_M|_{\mathcal{L}}}}$$

(can be defined perturbatively.)

Suppose for a moment that $\mathcal{F}_M$ is finite-dimensional and

Suppose for a moment that $\mathcal{F}_M^{\nu}$ is finite-dimensional and

$\omega_M = \sum dp_i \wedge dq_i$. [$\deg p_i + \deg q_i = -1$]

Then $\Delta = \sum_i \pm \underbrace{\frac{\partial}{\partial p_i}\frac{\partial}{\partial q_i}}$ satisfies ["BV operator"]

$$\Delta^2 = 0$$

$$\Delta(fg) = \pm(\Delta f)g \pm f(\Delta g) \pm \underline{(f, g)}.$$

<u>BV theorem</u> If $\boxed{\Delta(e^{i/\hbar S}) = 0}$ (Quantum Master Equation)

$$\Rightarrow \frac{d}{dt} \underbrace{\int_{\mathcal{L}_t} e^{i/\hbar S|_{\mathcal{L}_t}}}_{Z_t} = 0.$$ $\mathcal{L}_t$ family of lagrangians

Generalization: $\underset{\sim}{\mathcal{F}_M} = \underline{Y'} \times \underline{Y''}$ $\quad \underline{\omega_M} = \underline{\omega_{Y'}} + \underline{\omega_{Y''}}$

$$\Delta = \Delta_{Y'} + \Delta_{Y''}$$

$\mathcal{L} \subset Y''$ $\rightsquigarrow$ $\psi' = \int_{\mathcal{L} \subset Y''} e^{i/\hbar S|_{\mathcal{L}}}$ satisfies

$\Delta_{Y'} \psi' = 0$. $\psi' \in \underline{\mathcal{O}(Y')}$ called "<u>BV pushforward</u>"

$$\int_{Y''} : \mathcal{O}(\mathcal{F}_M) \longrightarrow \mathcal{O}(Y')$$

<u>BV-BFV formalism</u> | Batalin, Vilkovisky '77
| Batalin, Fradkin '83

On manifolds with boundary, Poisson bracket of local functionals is ill-defined due to boundary terms.

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Solution I: Fix boundary conditions $\to$ incompatible with gluing

Solution II: Relax BV to BV-BFV

$(\underbrace{\mathcal{F}_M, S_M, Q_M, \omega_M}_{})$ local $\to$ can define on mfld with boundary

Cattaneo-Mnev-Reshetikhin '2014:

$(\mathcal{F}_M, S_M, Q_M, \omega_M)$ induce* a "BFV structure" associated to ∂M:

*under some assumptions

$\mathcal{F}^\partial_{\partial M}$ - graded vector space

$\alpha^\partial_{\partial M} \in \Omega^1(\mathcal{F}^\partial_{\partial M})$ degree 0

$S^\partial_{\partial M} \in \mathcal{O}(\mathcal{F}^\partial_{\partial M})$ degree 1

Schiavina / Cattaneo $\to$ Gravity

s.t. $\begin{cases} \delta \alpha^\partial_{\partial M} = \omega^\partial_{\partial M} \text{ is symplectic} \\ (S^\partial_{\partial M}, S^\partial_{\partial M})_{\omega^\partial_{\partial M}} = 0. \end{cases}$

$\exists$ surjective submersion $\pi: \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$ and

$$\boxed{\delta S_M = \iota_{Q_M} \omega_M + \boxed{\pi^* \alpha^\partial_{\partial M}}}$$

BV-BFV compatibility relation

$$\Rightarrow \frac{1}{2} \iota_{Q_M} \iota_{Q_M} \omega_M = \boxed{\pi^* S^\partial_{\partial M}}$$

modified CME

Alexandrov-Kontsevich-Schwarz-Zaboronsky

CMR 2014: All AKSZ theories give [fully extended] BV-BFV theories.

Symmetry of the data: $f \in \mathcal{O}(\mathcal{F}^\partial_{\partial M})$

$\rightsquigarrow S^f = S + \pi^* f$

symmetry : $f \in C(\mathcal{F}^\partial_{\partial M})$
of the data

$$\leadsto \begin{cases} S_M^f = S_M + \pi^* f \\ \alpha_{\partial M}^{\partial, f} = \alpha_{\partial M}^{\partial} + \delta f \end{cases}$$

+ other data, still defines BV-BFV theory.

<u>Quantization</u> CMR 2018, 2020

I) $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M}, S^\partial_{\partial M})$ <u>Choose</u>: $\boxed{\mathcal{F}^\partial_{\partial M} = \mathcal{B} \times \mathcal{B}'}$ Lagrangian

$\leadsto (\mathcal{H}^{\mathcal{B}}_{\partial M}, \Omega^{\mathcal{B}}_{\partial M})$ "BFV complex" [Stasheff, Schätz]

$\mathcal{H}^{\mathcal{B}}_{\partial M} \simeq Fun(\mathcal{B})$, $\Omega^{\mathcal{B}}_{\partial M} \rightsquigarrow S^\partial_{\partial M}(b, -i\hbar \frac{\partial}{\partial b})$ + corrections

For $b \in \mathcal{B}$, want to compute $Z[b]$

II) Have $p : \mathcal{F}_M \xrightarrow{\pi} \mathcal{F}^\partial_{\partial M} \to \mathcal{B}$

<u>Choose</u> : · $0 \to \mathcal{Y} \to \mathcal{F}_M \underset{s}{\overset{p}{\rightleftarrows}} \mathcal{B} \to 0 \Rightarrow \mathcal{B} \times \mathcal{Y} \to \mathcal{F}_M$

· f s.t. $\pi^* \alpha^f|_{\mathcal{Y}} = 0 \Rightarrow$ "$\delta S = \iota_Q \omega$ on $\mathcal{Y}$"

· $\mathcal{Y} = \mathcal{Y}' \times \mathcal{Y}''$ s.t. $\exists \mathcal{L} \subset \mathcal{Y}''$, $S|_{\mathcal{L}}$ has nondegenerate crit pts.

"gauge fixing"

$$\boxed{Z_M(b, y') = \int_{y''\in\mathcal{L}\subset Y''} e^{\frac{i}{\hbar} S^{f}_{M}[b, y', y'']}} \in \mathrm{Fun}(B \times Y')[[\hbar]]$$

BV pushforward in family

In finite-dimensional model, have

$$\boxed{\left(\Delta^{BV}_{Y'} + \Omega^{B}_{BFV}\right) Z_M = 0}$$

"modified" QME

Gluing $Z_{M\cup_\Sigma M'} = \int_{B\times B' = \boxed{\tilde{B}_{\partial M}}} Z_M(b, y)\, Z_{M'}(b', \tilde{y})$

(perturbative integral)

Wishful thinking

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathrm{Cob} \arrow[r, "\text{Classical BV-BFV}", "(F_M, S_M)"'] \arrow[dr, dashed, "Z_{fun}"'] & \mathrm{BFV} \arrow[r, "\text{pert. quantization}"] & \mathrm{Ch}[[\hbar]] \arrow[d, "H^0"] \\
& \mathrm{Vect} \arrow[r, dashed, bend left, "\sim \hbar \to 0"] & \mathrm{Vect}[[\hbar]] \arrow[l, "\text{Resurgence?}"]
\end{tikzcd}$$

Ob BFV

Mor BV-BFV

(+ extended version...)

Some results in this direction.

Return to Chern-Simons theory.

M 3-mfld w/ boundary ∂M

$G \hookrightarrow P \twoheadrightarrow M$ principal G-bundle

Assume $P \simeq M \times G$ fixed

$$\begin{cases} F_M \cong \Omega^1(M, \mathfrak{g}) \\ S_M = \int_M \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A,A]\rangle \end{cases}$$

BV-BFV extension

bulk BV structure

superfield

$$\mathcal{F}_M = \Omega^\bullet(M, \mathfrak{g})[1] \ni \mathcal{A} \qquad \omega_M = \frac{1}{2}\int_M \langle \delta\mathcal{A}, \delta\mathcal{A}\rangle$$

$= \underset{0}{c} + \underset{1}{A} + \underset{2}{A^+} + \underset{3}{c^+}$ $\qquad (\delta c, \delta c^+) + \langle \delta A, \delta A^+\rangle$

$$S_M = \int_M \frac{1}{2}\langle \mathcal{A}, d\mathcal{A}\rangle + \frac{1}{6}\langle \mathcal{A}, [\mathcal{A},\mathcal{A}]\rangle$$

[only 3-form component]

$$Q_M = \frac{1}{2}\int_M \left\langle F_{\mathcal{A}}, \frac{\delta}{\delta\mathcal{A}}\right\rangle$$

BFV boundary structure

$$\mathcal{F}^\partial_{\partial M} = \Omega^\bullet(\partial M, \mathfrak{g})[1] \ni \mathcal{A}^\partial$$

- Abelian CS M. Manoliu '98
- QM T. Johnson-Freyd '10
- 1d CS Alekseev-Mnev '11
- 2d Yang-Mills Mnev-Iraso '19 (no functoriality, but corners)
- 2d scalar theory Kandel-Mnev-W. (functoriality)
  arXiv:1912.11202

$$\alpha^\partial_{\partial M} = \frac{1}{2}\int_{\partial M} \langle \mathcal{A}^\partial, \delta \mathcal{A}^\partial \rangle \qquad \omega^\partial = \int_{\partial M} \langle \delta c, \delta A^+ \rangle + \frac{1}{2}\int_{\partial M} \langle \delta A, \delta A \rangle$$

$$S^\partial_{\partial M} = \int_{\partial M} \frac{1}{2} \langle \mathcal{A}^\partial, d\mathcal{A}^\partial \rangle + \frac{1}{6} \langle \mathcal{A}^\partial, [\mathcal{A}^\partial, \mathcal{A}^\partial] \rangle$$

[only 2-form component]

<u>perturbative</u> <u>quantization</u> : Summary of results (ongoing work

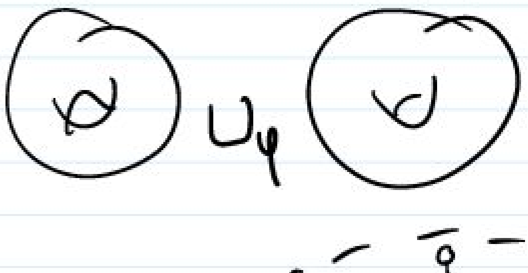
Step I) Find a Lagrangian splitting of
$\Omega^\bullet(\partial M, g)$ w.r.t $\int \langle \delta \mathcal{A}^\partial, \delta \mathcal{A}^\partial \rangle$

<u>two possibilities</u> : i) Find a lagrangian splitting $\boxed{\text{of } g}$ w.r.t
$\langle \cdot, \cdot \rangle$ $\quad g = V_1 \oplus V_2 \to$ split Chern-Simons

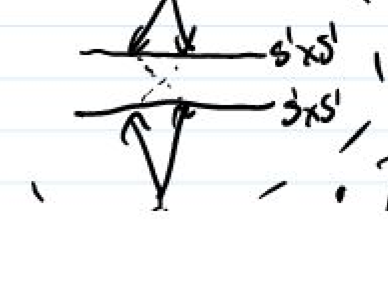
$$\Omega^\bullet(\partial M, g) = \underbrace{\Omega^\bullet(\partial M, V_1)}_{\mathcal{B}^\partial} \oplus \Omega^\bullet(\partial M, V_2)$$

<u>Some results</u> joint w/ Cattaneo, Mnev

Consider a lens space glued from two solid tori



- For V_1, V_2 subalgebras we recover known results for 2-loop approximation
[Casson invariant + φ-dependent term]



- If V_1, V_2 are not subalgebras then

• If V_1, V_2 are not subalgebras then behaviour is weird $\rightsquigarrow$ anomaly

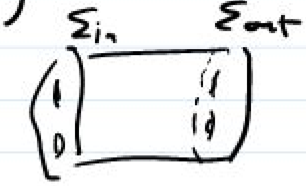
ii) Choose complex structure on $\partial M = \Sigma$ and write

$$\Omega^\bullet_{\mathbb{C}}(\partial M, \mathfrak{g}) = \underbrace{\Omega^0_{\mathbb{C}}(\partial M, \mathfrak{g}) \oplus \Omega^{0,1}(\partial M, \mathfrak{g})}_{\mathcal{B}} \oplus \Omega^{1,0}(\partial M, \mathfrak{g}) \oplus \Omega^2(\partial M, \mathfrak{g})$$

<u>results</u> • Alekseev–Mnev–Barmaz '13

in genus 0 : $H^0(\mathcal{H}^{\mathcal{B}}_{\partial M}, \Omega^{\mathcal{B}}) \cong$ conformal blocks
(w/ Wilson lines)

• Cattaneo–Mnev–W. (in progress)
quantization on $I \times \Sigma$



Interesting observation: for $\mathcal{B} = \underset{A^{0,1}_{in}}{\Omega^{0,1}(\Sigma_{in})} \oplus \underset{A^{1,0}}{\Omega^{1,0}(\Sigma_{out})} \oplus \underset{c_{in}}{\Omega^0(\Sigma_{in})} \oplus \underset{c_{out}}{\Omega^0(\Sigma_{out})}$

$$\mathcal{Y}' = \underset{\sigma}{\Omega^0(\partial M, \mathfrak{g})} \oplus \underset{A^+}{\Omega^2(\partial M, \mathfrak{g})}$$

we have

i/a S_{eff}

$$Z[A, c, \sigma, A^+] = e^{i/\hbar\, S_{eff}}$$

and

$$S_{eff}[A^{1,0}_{in}, A^{0,1}_{out}, c_{in}, c_{out}, \sigma, A^+]$$

$$= S_{WZW}[A^{0,1}_{in}, A^{0}_{out}, g] + \int_\Sigma \langle g^+, c_{in} - c_{out} \rangle$$

here $(\sigma, A^+) \rightsquigarrow (g, g^+)$ $\qquad g = \exp \sigma \in \Omega^0(\Sigma, G)$

and $\boxed{(\Delta_y + \Omega_\hbar) Z = 0}$ $\Rightarrow$ Polyakov-Wiegmann identity for S_{WZW}

<u>Open problems</u>:

- Complex bdry conditions for arbitrary Σ
- $H^0(\Sigma, \Omega_\Sigma) \simeq$ conformal blocks $(g>0)$
- Corners and MTCs
- Gluing of more general 3-mflds, comparison to Z_{RT}