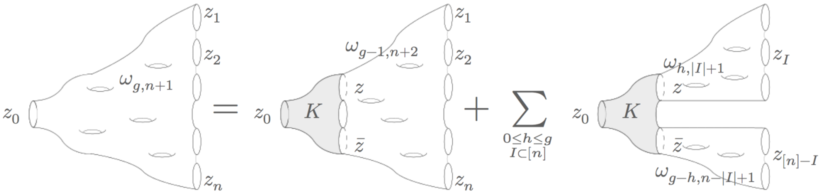
Enumerative algebraic geometry, topological recursion, and integrable hierarchies

Danilo Lewański
IHES and IPhT, Paris

November the 17th, 2020

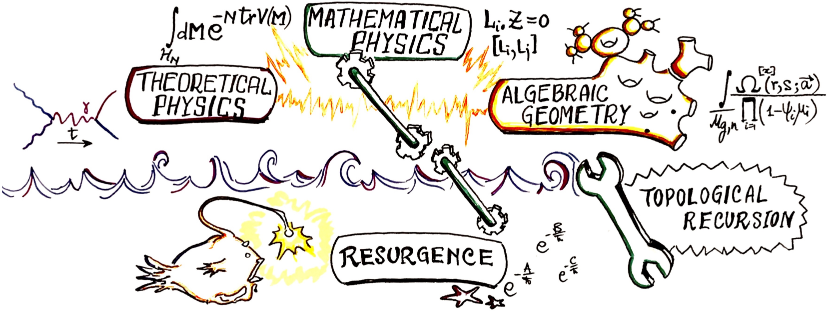


$$\text{Spectral Curve} \xrightarrow{\text{Topological recursion}} \text{Invariants } \omega_{g,n}.$$

Definition

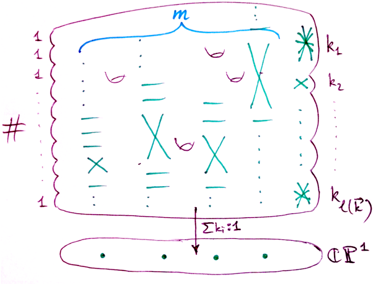
Let us define ENT the class of enumerative problems for which there exists a proof that the solution can be generated by TR and/or ABCD-TR and/or GR, for some initial data.

I am interested in expanding ENT as well as in the following picture for ENT:



- Algebraic geometry: cohomological field theories, moduli spaces of curves, ...
- Mathematical physics: integrable hierarchies, ...
- Theoretical physics (GW / top strings / ...much more) or statistical physics (matrix models) or any motivation for the enumerative problem, ...

An example: Hurwitz numbers



Definition (Hurwitz numbers)

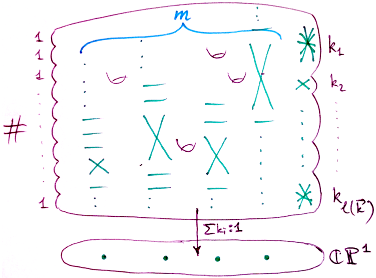
For a partition α of size d, let $C_\alpha \in \mathbb{Q}[\mathfrak{S}_d]$ be the formal sum of all permutations in $\mathfrak{S}_d$ of cycle type α. For a non-negative integer g and a partition k of d of length n define

$$h^{\bullet}_{g;k} := \frac{1}{d!}[C_{id}].C_k \frac{C_{(2)}^m}{m!} C_{(1,1,\dots,1)}, \qquad h^{\circ}_{g;\vec{k}} := \frac{1}{d!}[C_{id}]^{\circ}.C_k \frac{C_{(2)}^m}{m!} C_{(1,1,\dots,1)}$$

where $m = 2g - 2 + n + d$. $[C_{id}]^{\circ}$ only counts the products of tuples generating transitive subgroups. Equivalently, $h^{\bullet}_{g;k}$ and $h^{\circ}_{g;\vec{k}}$ are related by inclusion-exclusion.

- They enumerate branched covers with prescribed ramification conditions.
- They enumerate constellations by lifting the graph passing through all branch points.

An example: Hurwitz numbers



Definition (Hurwitz numbers)

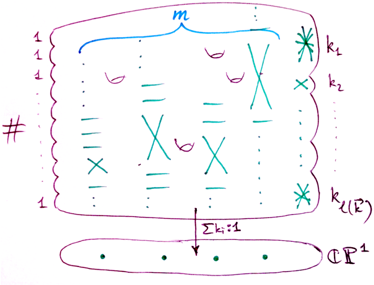
For a partition α of size d, let $C_\alpha \in \mathbb{Q}[\mathfrak{S}_d]$ be the formal sum of all permutations in $\mathfrak{S}_d$ of cycle type α. For a non-negative integer g and a partition k of d of length n define

$$h^{\bullet}_{g;k} := \frac{1}{d!}[C_{id}].C_k \frac{C_{(2)}^m}{m!} C_{(1,1,\dots,1)}, \qquad h^{\circ}_{g;\vec{k}} := \frac{1}{d!}[C_{id}]^{\circ}.C_k \frac{C_{(2)}^m}{m!} C_{(1,1,\dots,1)}$$

where $m = 2g - 2 + n + d$. $[C_{id}]^{\circ}$ only counts the products of tuples generating transitive subgroups. Equivalently, $h^{\bullet}_{g;k}$ and $h^{\circ}_{g;\vec{k}}$ are related by inclusion-exclusion.

- They enumerate branched covers with prescribed ramification conditions.
- They enumerate constellations by lifting the graph passing through all branch points.

An example: Hurwitz numbers



Definition (Hurwitz numbers)

For a partition α of size d, let $C_\alpha \in \mathbb{Q}[\mathfrak{S}_d]$ be the formal sum of all permutations in $\mathfrak{S}_d$ of cycle type α. For a non-negative integer g and a partition k of d of length n define

$$h^{\bullet}_{g;k} := \frac{1}{d!}[C_{id}].C_k \frac{C_{(2)}^m}{m!} C_{(1,1,\dots,1)}, \qquad h^{\circ}_{g;\vec{k}} := \frac{1}{d!}[C_{id}]^{\circ}.C_k \frac{C_{(2)}^m}{m!} C_{(1,1,\dots,1)}$$

where $m = 2g - 2 + n + d$. $[C_{id}]^{\circ}$ only counts the products of tuples generating transitive subgroups. Equivalently, $h^{\bullet}_{g;k}$ and $h^{\circ}_{g;\vec{k}}$ are related by inclusion-exclusion.

- They enumerate branched covers with prescribed ramification conditions.
- They enumerate constellations by lifting the graph passing through all branch points.

Algebro-geometric side

In terms of the intersection theory of the moduli space of curves:

Theorem (Ekedahl, Lando, Shapiro, Vainshtein ('99))

Let g, n be non-negative integers such that $2g-2+n>0$. For a partition k of length n and size d we have :

$$h^{\circ}_{g;k} = \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{(c(\mathbb{E}^{\vee}) = 1 - \lambda_1 + \ldots + (-1)^g \lambda_g)}{\prod_{i=1}^{n}(1 - k_i \psi_i)}$$

where:

- $h^{\circ}_{g;k} = [C_{id}]^{\circ} . C_k \frac{(C_{(2)})^m}{m!} C_{(1,1,\ldots,1)}$
- By Riemann Hurwitz $m = (2g-2+n+d)$
- $\mathbb{E}$ is the Hodge bundle

Theorem (Mumford formula)

$$c(\mathbb{E}^{\vee}) = \exp\left(\sum_{m=1} (-1)^m \frac{B_{m+1}}{m(m+1)} \left[\kappa_m - \sum_{i=1}^{n} \psi_i^m + \frac{1}{2} j_* \frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}\right]\right)$$

where j is the boundary morphism representing the boundary divisor at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, B_m are Bernoulli numbers.

Algebro-geometric side

In terms of the intersection theory of the moduli space of curves:

Theorem (Ekedahl, Lando, Shapiro, Vainshtein ('99))

Let g, n be non-negative integers such that $2g - 2 + n > 0$. For a partition k of length n and size d we have :

$$h^{\circ}_{g;k} = \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{(c(\mathbb{E}^{\vee}) = 1 - \lambda_1 + \ldots + (-1)^g \lambda_g)}{\prod_{i=1}^{n}(1 - k_i \psi_i)}$$

where:

- $h^{\circ}_{g;k} = [C_{id}]^{\circ} . C_k \frac{(C_{(2)})^m}{m!} C_{(1,1,\ldots,1)}$
- By Riemann Hurwitz $m = (2g - 2 + n + d)$
- $\mathbb{E}$ is the Hodge bundle

Theorem (Mumford formula)

$$c(\mathbb{E}^{\vee}) = \exp\left(\sum_{m=1} (-1)^m \frac{B_{m+1}}{m(m+1)} \left[\kappa_m - \sum_{i=1}^{n} \psi_i^m + \frac{1}{2} j_* \frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}\right]\right)$$

where j is the boundary morphism representing the boundary divisor at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, B_m are Bernoulli numbers.

Mathematical physics side

In terms of integrable hierarchies:

Theorem (Okounkov, '00)

The partition function of double Hurwitz numbers

$$\mathcal{Z}(p, \beta; \vec{p}, \vec{p}') := \sum_{d,m;\mu,\nu\vdash d} h^{\bullet}_{g;\mu,\nu} q^d \beta^m p_\mu, p'_\nu, \qquad h^{\bullet}_{g;\mu,\nu} = \frac{1}{d!}[C_{id}].C_\mu \frac{C^m_{(2)}}{m!} C_\nu$$

is a tau-function of the 2D Toda integrable hierarchy.

Corollary (Okounkov, '00)

The partition function of Hurwitz numbers is a tau-function of the KP integrable hierarchy.

- It extends to many other Hurwitz problems (different ramification conditions)
- It proves a recursion for Hurwitz numbers conjectured by Pandharipande
- The key part of the proof is to express (changing to Schur functions basis)

$$h^{\bullet}_{g;\mu,\nu} = \frac{1}{\prod \mu_i \prod \nu_j} \cdot \left\langle \alpha_{\mu_1}\alpha_{\mu_2}\cdots\alpha_{\mu_{\ell(\mu)}} \frac{\mathcal{F}_2\mathcal{F}_2\cdots\mathcal{F}_2}{m!} \alpha_{-\nu_1}\alpha_{-\nu_2}\cdots\alpha_{-\nu_{\ell(\nu)}} \right\rangle$$

for the bosonic operators α_k, $[\alpha_k, \alpha_l] = k\delta_{k,l}$ acting on the Fock space and some zero energy operators $\mathcal{F}_2$ representing simple ramifications.

Mathematical physics side
In terms of integrable hierarchies:

Theorem (Okounkov, '00)

The partition function of double Hurwitz numbers

$$\mathcal{Z}(p,\beta;\vec{p},\vec{p}') := \sum_{d,m;\mu,\nu\vdash d} h^{\bullet}_{g;\mu,\nu} q^d \beta^m p_\mu, p'_\nu, \qquad h^{\bullet}_{g;\mu,\nu} = \frac{1}{d!}[C_{id}].C_\mu \frac{C^m_{(2)}}{m!} C_\nu$$

is a tau-function of the 2D Toda integrable hierarchy.

Corollary (Okounkov, '00)

The partition function of Hurwitz numbers is a tau-function of the KP integrable hierarchy.

- It extends to many other Hurwitz problems (different ramification conditions)
- It proves a recursion for Hurwitz numbers conjectured by Pandharipande
- The key part of the proof is to express (changing to Schur functions basis)

$$h^{\bullet}_{g;\mu,\nu} = \frac{1}{\prod \mu_i \prod \nu_j} \cdot \left\langle \alpha_{\mu_1}\alpha_{\mu_2}\cdots\alpha_{\mu_{\ell(\mu)}} \frac{\mathcal{F}_2\mathcal{F}_2\cdots\mathcal{F}_2}{m!} \alpha_{-\nu_1}\alpha_{-\nu_2}\cdots\alpha_{-\nu_{\ell(\nu)}} \right\rangle$$

for the bosonic operators α_k, $[\alpha_k,\alpha_l] = k\delta_{k,l}$ acting on the Fock space and some zero energy operators $\mathcal{F}_2$ representing simple ramifications.

Mathematical physics side
In terms of integrable hierarchies:

Theorem (Okounkov, '00)

The partition function of double Hurwitz numbers

$$\mathcal{Z}(p,\beta;\vec{p},\vec{p}') := \sum_{d,m;\mu,\nu\vdash d} h^{\bullet}_{g;\mu,\nu} q^d \beta^m p_\mu, p'_\nu, \qquad h^{\bullet}_{g;\mu,\nu} = \frac{1}{d!}[C_{id}].C_\mu \frac{C^m_{(2)}}{m!} C_\nu$$

is a tau-function of the 2D Toda integrable hierarchy.

Corollary (Okounkov, '00)

The partition function of Hurwitz numbers is a tau-function of the KP integrable hierarchy.

- It extends to many other Hurwitz problems (different ramification conditions)
- It proves a recursion for Hurwitz numbers conjectured by Pandharipande
- The key part of the proof is to express (changing to Schur functions basis)

$$h^{\bullet}_{g;\mu,\nu} = \frac{1}{\prod \mu_i \prod \nu_j} \cdot \left\langle \alpha_{\mu_1}\alpha_{\mu_2}\cdots\alpha_{\mu_{\ell(\mu)}} \frac{\mathcal{F}_2\mathcal{F}_2\cdots\mathcal{F}_2}{m!} \alpha_{-\nu_1}\alpha_{-\nu_2}\cdots\alpha_{-\nu_{\ell(\nu)}} \right\rangle$$

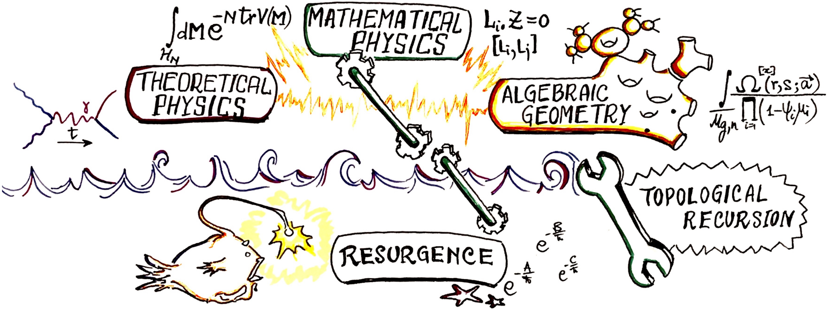
for the bosonic operators α_k, $[\alpha_k,\alpha_l] = k\delta_{k,l}$ acting on the Fock space and some zero energy operators $\mathcal{F}_2$ representing simple ramifications.

Physics/enumerative problem side

In terms of enumerative problems / topological string theory / matrix models:

- Gromov Witten / Hurwitz correspondence for nonsingular target curves (Okounkov-Pandharipande, '02)
- BKMP conjecture (Bouchard, Klemm, Mariño, Pasquetti, '07)
- Bouchard Mariño conjecture, '07 (mirrors of toric Calabi-Yau threefolds, framed vertex in the limit of infinite framing)
- Matrix model for Hurwitz numbers (Borot, Eynard, Mulase, Safnuk, '09)

What about topological recursion for Hurwitz theory?



Topological recursion for Hurwitz numbers

Conjecture (Bouchard-Mariño, '07. Now theorem)

Hurwitz numbers belong to ENT via the spectral curve

$$\Sigma = \mathbb{CP}^1, \quad x(z) = -z + \log(z), \quad y(z) = z, \quad \omega_{0,2} = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

producing free energies $F_{g,n}(x_1, \ldots, x_n) = \int^{x_1} \cdots \int^{x_n} \omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1} h^{\circ}_{g;\mu} \prod_{i=1}^{n} e^{x_i \mu_i}$

Theorem (Eynard, '11; DOSS, '13)

$$\omega_{g,n}(\vec{x}) = \sum_{\substack{a_1,\ldots,a_n \\ d_1,\ldots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} \mathcal{C}_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \prod_{i=1}^{n} \psi_i^{d_i} \mathrm{d}\left(\left(-\frac{d}{dx_i}\right)^{d_i} \xi_{a_i}(x_i)\right)$$

- (Eynard, Mulase, Safnuk, '09): Proof using ELSV formula
- For Hurwitz numbers we have $\xi_0(x) = \sum_{\mu=0} \frac{\mu^\mu}{\mu!} e^{\mu x}$, which gives exactly the non-polynomial part of the ELSV formula
- The ELSV formula is implied to extract the non-polynomial structure, to show that the correlation differentials are well-defined on the spectral curve as they can be expressed as differential operators acting on the basis of the ξ_i.

Topological recursion for Hurwitz numbers

Conjecture (Bouchard-Mariño, '07. Now theorem)

Hurwitz numbers belong to ENT via the spectral curve

$$\Sigma = \mathbb{CP}^1, \quad x(z) = -z + \log(z), \quad y(z) = z, \quad \omega_{0,2} = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

producing free energies $F_{g,n}(x_1, \ldots, x_n) = \int^{x_1} \cdots \int^{x_n} \omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1} h^{\circ}_{g;\mu} \prod_{i=1}^{n} e^{x_i \mu_i}$

Theorem (Eynard, '11; DOSS, '13)

$$\omega_{g,n}(\vec{x}) = \sum_{\substack{a_1,\ldots,a_n \\ d_1,\ldots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} \mathcal{C}_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \prod_{i=1}^{n} \psi_i^{d_i} \mathrm{d}\left(\left(-\frac{d}{dx_i}\right)^{d_i} \xi_{a_i}(x_i)\right)$$

- (Eynard, Mulase, Safnuk, '09): Proof using ELSV formula
- For Hurwitz numbers we have $\xi_0(x) = \sum_{\mu=0} \frac{\mu^\mu}{\mu!} e^{\mu x}$, which gives exactly the non-polynomial part of the ELSV formula
- The ELSV formula is implied to extract the non-polynomial structure, to show that the correlation differentials are well-defined on the spectral curve as they can be expressed as differential operators acting on the basis of the ξ_i.

Topological recursion for Hurwitz numbers

Conjecture (Bouchard-Mariño, '07. Now theorem)

Hurwitz numbers belong to ENT via the spectral curve

$$\Sigma = \mathbb{CP}^1, \quad x(z) = -z + \log(z), \quad y(z) = z, \quad \omega_{0,2} = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

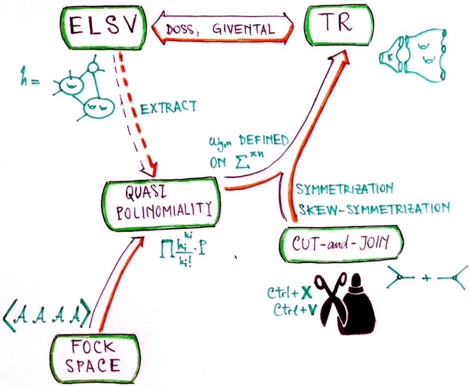
producing free energies $F_{g,n}(x_1, \ldots, x_n) = \int^{x_1} \cdots \int^{x_n} \omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1} h^{\circ}_{g;\mu} \prod_{i=1}^{n} e^{x_i \mu_i}$

Theorem (Eynard, '11; DOSS, '13)

$$\omega_{g,n}(\vec{x}) = \sum_{\substack{a_1,\ldots,a_n \\ d_1,\ldots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} \mathcal{C}_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \prod_{i=1}^{n} \psi_i^{d_i} \mathrm{d}\left(\left(-\frac{d}{dx_i}\right)^{d_i} \xi_{a_i}(x_i)\right)$$

- (Eynard, Mulase, Safnuk, '09): Proof using ELSV formula
- For Hurwitz numbers we have $\xi_0(x) = \sum_{\mu=0} \frac{\mu^\mu}{\mu!} e^{\mu x}$, which gives exactly the non-polynomial part of the ELSV formula
- The ELSV formula is implied to extract the non-polynomial structure, to show that the correlation differentials are well-defined on the spectral curve as they can be expressed as differential operators acting on the basis of the ξ_i.

TR for algebraic geometry: proving ELSV formulae by extracting the non-polynomial part independently



- Example: new proof of Johnson-Pandharipande-Tseng ELSV formula for orbifold Hurwitz numbers.

Definition

- Let g, n be non-negative integers such that $2g-2+n>0$.
- Let $r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$.
- Let $\{a_1, \ldots, a_n\} \in [0, r-1]^n$ an integer vector such that: $\sum_i a_i \equiv s(2g-2+n) \pmod{r}$
- Let $\overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the proper moduli stack of stable curves $[C, p_1, \ldots, p_n]$ in $\overline{\mathcal{M}}_{g,n}$ together with a line bundle L such that $L^{\otimes r} \cong \omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$
- Let $\pi : \overline{\mathcal{C}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the universal curve, let $\mathcal{L} \to \overline{\mathcal{C}}^{r,s}_{g,n,a}$ be the r-th universal root, let $\epsilon : \overline{\mathcal{M}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful map. Let $\mathrm{ch}_m(r, s; \vec{a})$ be the Chern character $\mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})$

Theorem (Chiodo formula)

$$(m+1)!\,\mathrm{ch}_m(r, s; \vec{a}) = B_{m+1}\left(\frac{s}{r}\right)\kappa_m - \sum_{i=1}^{n} B_{m+1}\left(\frac{a_i}{r}\right)\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right) j_{a,*}\frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}$$

where j_a is the boundary morphism representing the boundary divisor with multiplicity a at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, $B_m(x)$ are Bernoulli polynomials.

Definition (Chiodo classes)

$$\Omega^{[x]}(r, s; \vec{a}) := \epsilon_* \exp\left(\sum_{m=1}(-x)^m (m-1)!\,\mathrm{ch}_m(r, s; \vec{a})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$$

Definition

- Let g, n be non-negative integers such that $2g - 2 + n > 0$.
- Let $r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$.
- Let $\{a_1, \dots, a_n\} \in [0, r-1]^n$ an integer vector such that: $\sum_i a_i \equiv s(2g-2+n) \pmod r$
- Let $\overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the proper moduli stack of stable curves $[C, p_1, \dots, p_n]$ in $\overline{\mathcal{M}}_{g,n}$ together with a line bundle L such that $L^{\otimes r} \cong \omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$
- Let $\pi : \overline{\mathcal{C}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the universal curve, let $\mathcal{L} \to \overline{\mathcal{C}}^{r,s}_{g,n,a}$ be the r-th universal root, let $\epsilon : \overline{\mathcal{M}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful map. Let $\mathrm{ch}_m(r, s; \vec{a})$ be the Chern character $\mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})$

Theorem (Chiodo formula)

$$(m+1)!\,\mathrm{ch}_m(r,s;\vec{a}) = B_{m+1}\left(\frac{s}{r}\right)\kappa_m - \sum_{i=1}^{n} B_{m+1}\left(\frac{a_i}{r}\right)\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right) j_{a,*}\frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}$$

where j_a is the boundary morphism representing the boundary divisor with multiplicity a at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, $B_m(x)$ are Bernoulli polynomials.

Definition (Chiodo classes)

$$\Omega^{[x]}(r,s;\vec{a}) := \epsilon_* \exp\left(\sum_{m=1}(-x)^m (m-1)!\,\mathrm{ch}_m(r,s;\vec{a})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$$

Definition

- Let g, n be non-negative integers such that $2g - 2 + n > 0$.
- Let $r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$.
- Let $\{a_1, \dots, a_n\} \in [0, r-1]^n$ an integer vector such that: $\sum_i a_i \equiv s(2g-2+n) \pmod r$
- Let $\overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the proper moduli stack of stable curves $[C, p_1, \dots, p_n]$ in $\overline{\mathcal{M}}_{g,n}$ together with a line bundle L such that $L^{\otimes r} \cong \omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$
- Let $\pi : \overline{\mathcal{C}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the universal curve, let $\mathcal{L} \to \overline{\mathcal{C}}^{r,s}_{g,n,a}$ be the r-th universal root, let $\epsilon : \overline{\mathcal{M}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful map. Let $\mathrm{ch}_m(r,s;\vec{a})$ be the Chern character $\mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})$

Theorem (Chiodo formula)

$$(m+1)!\,\mathrm{ch}_m(r,s;\vec{a}) = B_{m+1}\left(\frac{s}{r}\right)\kappa_m - \sum_{i=1}^{n} B_{m+1}\left(\frac{a_i}{r}\right)\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right) j_{a,*}\frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}$$

where j_a is the boundary morphism representing the boundary divisor with multiplicity a at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, $B_m(x)$ are Bernoulli polynomials.

Definition (Chiodo classes)

$$\Omega^{[x]}(r,s;\vec{a}) := \epsilon_* \exp\left(\sum_{m=1}(-x)^m (m-1)!\,\mathrm{ch}_m(r,s;\vec{a})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$$

Definition

- Let g, n be non-negative integers such that $2g-2+n>0$.
- Let $r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$.
- Let $\{a_1, \ldots, a_n\} \in [0, r-1]^n$ an integer vector such that: $\sum_i a_i \equiv s(2g-2+n) \pmod{r}$
- Let $\overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the proper moduli stack of stable curves $[C, p_1, \ldots, p_n]$ in $\overline{\mathcal{M}}_{g,n}$ together with a line bundle L such that $L^{\otimes r} \cong \omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$
- Let $\pi : \overline{\mathcal{C}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the universal curve, let $\mathcal{L} \to \overline{\mathcal{C}}^{r,s}_{g,n,a}$ be the r-th universal root, let $\epsilon : \overline{\mathcal{M}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful map. Let $\mathrm{ch}_m(r,s;\vec{a})$ be the Chern character $\mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})$

Theorem (Chiodo formula)

$$(m+1)!\,\mathrm{ch}_m(r,s;\vec{a}) = B_{m+1}\left(\frac{s}{r}\right)\kappa_m - \sum_{i=1}^{n} B_{m+1}\left(\frac{a_i}{r}\right)\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right) j_{a,*}\frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}$$

where j_a is the boundary morphism representing the boundary divisor with multiplicity a at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, $B_m(x)$ are Bernoulli polynomials.

Definition (Chiodo classes)

$$\Omega^{[x]}(r,s;\vec{a}) := \epsilon_* \exp\left(\sum_{m=1} (-x)^m (m-1)!\,\mathrm{ch}_m(r,s;\vec{a})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$$

Definition

- Let g, n be non-negative integers such that $2g - 2 + n > 0$.
- Let $r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$.
- Let $\{a_1, \ldots, a_n\} \in [0, r-1]^n$ an integer vector such that: $\sum_i a_i \equiv s(2g - 2 + n) \pmod{r}$
- Let $\overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the proper moduli stack of stable curves $[C, p_1, \ldots, p_n]$ in $\overline{\mathcal{M}}_{g,n}$ together with a line bundle L such that $L^{\otimes r} \cong \omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$
- Let $\pi : \overline{\mathcal{C}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the universal curve, let $\mathcal{L} \to \overline{\mathcal{C}}^{r,s}_{g,n,a}$ be the r-th universal root, let $\epsilon : \overline{\mathcal{M}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful map. Let $\mathrm{ch}_m(r, s; \vec{a})$ be the Chern character $\mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})$

Theorem (Chiodo formula)

$$(m+1)!\,\mathrm{ch}_m(r, s; \vec{a}) = B_{m+1}\left(\frac{s}{r}\right)\kappa_m - \sum_{i=1}^{n} B_{m+1}\left(\frac{a_i}{r}\right)\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right) j_{a,*}\frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}$$

where j_a is the boundary morphism representing the boundary divisor with multiplicity a at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, $B_m(x)$ are Bernoulli polynomials.

Definition (Chiodo classes)

$$\Omega^{[x]}(r, s; \vec{a}) := \epsilon_* \exp\left(\sum_{m=1}(-x)^m (m-1)!\,\mathrm{ch}_m(r, s; \vec{a})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$$

Definition

- Let g, n be non-negative integers such that $2g - 2 + n > 0$.
- Let $r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$.
- Let $\{a_1, \ldots, a_n\} \in [0, r-1]^n$ an integer vector such that: $\sum_i a_i \equiv s(2g-2+n) \pmod r$
- Let $\overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the proper moduli stack of stable curves $[C, p_1, \ldots, p_n]$ in $\overline{\mathcal{M}}_{g,n}$ together with a line bundle L such that $L^{\otimes r} \cong \omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$
- Let $\pi : \overline{\mathcal{C}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the universal curve, let $\mathcal{L} \to \overline{\mathcal{C}}^{r,s}_{g,n,a}$ be the r-th universal root, let $\epsilon : \overline{\mathcal{M}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful map. Let $\mathrm{ch}_m(r, s; \vec{a})$ be the Chern character $\mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})$

Theorem (Chiodo formula)

$$(m+1)!\,\mathrm{ch}_m(r,s;\vec{a}) = B_{m+1}\left(\frac{s}{r}\right)\kappa_m - \sum_{i=1}^{n} B_{m+1}\left(\frac{a_i}{r}\right)\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right) j_{a,*}\frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}$$

where j_a is the boundary morphism representing the boundary divisor with multiplicity a at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, $B_m(x)$ are Bernoulli polynomials.

Definition (Chiodo classes)

$$\Omega^{[x]}(r,s;\vec{a}) := \epsilon_* \exp\left(\sum_{m=1}(-x)^m (m-1)!\,\mathrm{ch}_m(r,s;\vec{a})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$$

Definition

- Let g, n be non-negative integers such that $2g-2+n>0$.
- Let $r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$.
- Let $\{a_1,\ldots,a_n\} \in [0,r-1]^n$ an integer vector such that: $\sum_i a_i \equiv s(2g-2+n) \pmod r$
- Let $\overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the proper moduli stack of stable curves $[C,p_1,\ldots,p_n]$ in $\overline{\mathcal{M}}_{g,n}$ together with a line bundle L such that $L^{\otimes r} \cong \omega_{\log}^{\otimes s}(-\sum_i a_i p_i)$
- Let $\pi : \overline{\mathcal{C}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}^{r,s}_{g,n,a}$ be the universal curve, let $\mathcal{L} \to \overline{\mathcal{C}}^{r,s}_{g,n,a}$ be the r-th universal root, let $\epsilon : \overline{\mathcal{M}}^{r,s}_{g,n,a} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful map. Let $\mathrm{ch}_m(r,s;\vec{a})$ be the Chern character $\mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})$

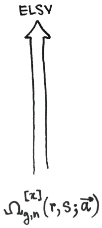
Theorem (Chiodo formula)

$$(m+1)!\,\mathrm{ch}_m(r,s;\vec{a}) = B_{m+1}\left(\frac{s}{r}\right)\kappa_m - \sum_{i=1}^{n} B_{m+1}\left(\frac{a_i}{r}\right)\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right) j_{a,*}\frac{(\psi')^m-(-\psi'')^m}{\psi'+\psi''}$$

where j_a is the boundary morphism representing the boundary divisor with multiplicity a at one of the branches of the node, ψ', ψ'' are the ψ classes at the branches of the node, $B_m(x)$ are Bernoulli polynomials.

Definition (Chiodo classes)

$$\Omega^{[x]}(r,s;\vec{a}) := \epsilon_* \exp\left(\sum_{m=1}(-x)^m (m-1)!\,\mathrm{ch}_m(r,s;\vec{a})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$$


ELSV
$\Omega_{g,n}^{[x]}(r,s;\vec{a})$

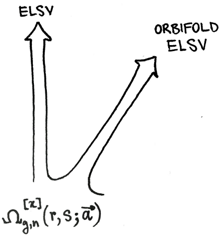
Theorem (Ekedahl, Lando, Shapiro, Vainshtein ('99))

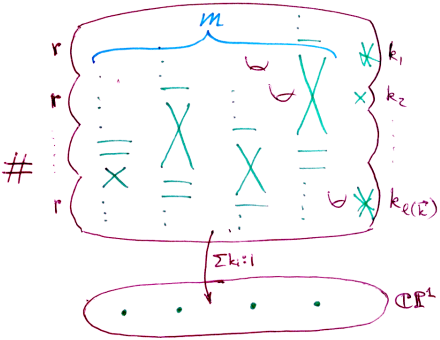
Let g, n be non-negative integers such that $2g-2+n>0$. For a partition μ of length n and size d we have :

$$h^{\circ}_{g;\mu} = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}(r=1, s=1; \vec{1})}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}$$

where:

- $h^{\circ}_{g;\mu} = \frac{1}{d!}[C_{id}]^{\circ}.C_{\mu}\frac{(C_{(2)})^m}{m!}C_{(1,1,\ldots,1)}$
- By Riemann Hurwitz $m=(2g-2+n+d)$


ELSV
ORBIFOLD ELSV
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$


m
r
r
r
#
k_1
k_2
$k_{\ell(\vec{k})}$
$\Sigma k_i : 1$
$\mathbb{CP}^1$

Proposition (L,Popolitov, Shadrin, Zvonkine, '15)

Johnson-Pandharipande-Tseng ELSV formula can be restated in terms of Chiodo class (slightly specialised) as follows.

Let g, n be non-negative integers such that $2g-2+n>0$. Let q be a positive integer. For a partition μ of length n and size d divisible by q, we have :

$$h^{\circ,q}_{g;\mu} = c^{q}_{g,n} \cdot \prod_{i=1}^{n} \frac{(\frac{\mu_i}{q})^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}_{g,n}(q,q;q-\langle\mu_i\rangle)}{\prod_{i=1}^{n}(1-\frac{\mu_i}{q}\psi_i)}$$

where:

- $h^{\circ,q}_{g;\mu} = [C_{id}]^{\circ}.C_{\mu}\frac{(C_{(2)})^m}{m!}C_{(q,q,\ldots,q)}$
- $\mu_i = q[\mu_i]+\langle\mu_i\rangle$, and by Riemann Hurwitz $m = (2g-2+\ell(\mu)+|\mu|/q)$
- $c^{q}_{g,n}$ is the product of powers of q depending on g, n, μ.

Theorem (L.,Popolitov, Shadrin, Zvonkine, '15)

The spectral curve $\Sigma = \mathbb{CP}^1$, $x=-z^r+log(z)$, $y=z^s$, $\omega_{0,2}=\frac{dz_1\,dz_2}{(z_1-z_2)^2}$ *produces free energies*

$$F^{r,s}_{g,n}(x_1,\ldots,x_n) = c^{r,s}_{g,n}\prod_{i=1}^{n}\frac{(\mu_i/r)^{[\mu_i]}}{[\mu_i]!}e^{x_i\mu_i}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Omega^{[1]}_{g,n}(r,s;r-\langle\mu_i\rangle)}{\prod_{i=1}^{n}(1-\frac{\mu_i}{r}\psi_i)}, \qquad \mu=[\mu]r+\langle\mu\rangle$$

Proof: DOSS equivalence (Givental/Teleman classification and TR).

Proposition (L,Popolitov, Shadrin, Zvonkine, '15)

Johnson-Pandharipande-Tseng ELSV formula can be restated in terms of Chiodo class (slightly specialised) as follows.

Let g, n be non-negative integers such that $2g-2+n>0$. Let q be a positive integer. For a partition μ of length n and size d divisible by q, we have :

$$h^{\circ,q}_{g;\mu} = c^q_{g,n} \cdot \prod_{i=1}^{n} \frac{(\frac{\mu_i}{q})^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}_{g,n}(q,q;q-\langle\mu_i\rangle)}{\prod_{i=1}^{n}(1-\frac{\mu_i}{q}\psi_i)}$$

where:

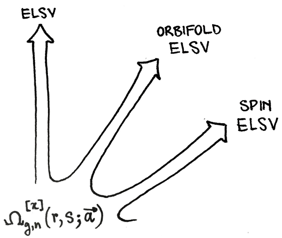
- $h^{\circ,q}_{g;\mu} = [C_{id}]^\circ . C_\mu \frac{(C_{(2)})^m}{m!} C_{(q,q,\ldots,q)}$
- $\mu_i = q[\mu_i] + \langle\mu_i\rangle$, and by Riemann Hurwitz $m = (2g-2+\ell(\mu)+|\mu|/q)$
- $c^q_{g,n}$ is the product of powers of q depending on g, n, μ.

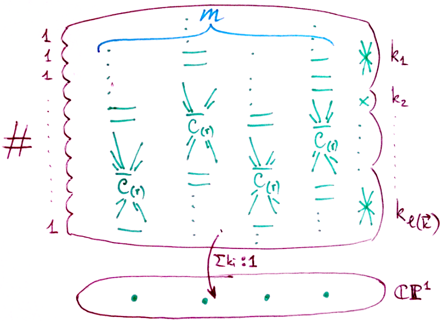
Theorem (L.,Popolitov, Shadrin, Zvonkine, '15)

The spectral curve $\Sigma = \mathbb{CP}^1$, $x = -z^r + log(z)$, $y = z^s$, $\omega_{0,2} = \frac{dz_1\, dz_2}{(z_1-z_2)^2}$ *produces free energies*

$$F^{r,s}_{g,n}(x_1,\ldots,x_n) = c^{r,s}_{g,n} \prod_{i=1}^{n} \frac{(\mu_i/r)^{[\mu_i]}}{[\mu_i]!} e^{x_i\mu_i} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}_{g,n}(r,s;r-\langle\mu_i\rangle)}{\prod_{i=1}^{n}(1-\frac{\mu_i}{r}\psi_i)}, \qquad \mu = [\mu]r + \langle\mu\rangle$$

Proof: DOSS equivalence (Givental/Teleman classification and TR).


ELSV
ORBIFOLD ELSV
SPIN ELSV
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$


m
1
1
1
1
#
$C_{(r)}$
$C_{(r)}$
$C_{(r)}$
$C_{(r)}$
k_1
k_2
$k_{\ell(\vec{k})}$
$\Sigma k_i : 1$
$\mathbb{CP}^1$

Conjecture (Zvonkine (unpublished, '06). Now theorem.)

Let g, n be non-negative integers such that $2g-2+n>0$. Let r be a positive integer. For a partition μ of length n and size d, we have :

$$h^{\circ, r-spin}_{g;\mu} = c^r_{g,n} \cdot \prod_{i=1}^{n} \frac{(\frac{\mu_i}{r})^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}_{g,n}(r, 1; r - \langle \mu_i \rangle)}{\prod_{i=1}^{n}(1 - \frac{\mu_i}{r}\psi_i)}$$

where:

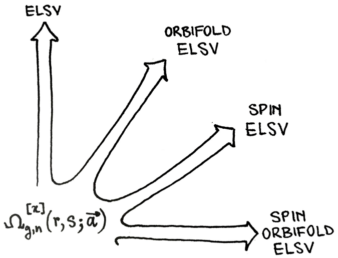
- $h^{\circ, r-spin}_{g;\mu} = [C_{id}]^{\circ}.C_{\mu} \frac{(\overline{C}_{r+1})^b}{b!} C_{(1,1,\dots,1)}$
- $\mu_i = r[\mu_i] + \langle \mu_i \rangle$, and by Riemann Hurwitz $b = (2g-2+n+d)/r$
- $c^r_{g,n}$ is the product of powers of r depending on g, n, μ.
- $\overline{C}_{r+1}$ is the $(r+1)$-st completed cycle of the GW/Hurwitz correspondence. For instance:

  $$\overline{C}_{(2)} = C_{(2)}, \qquad \overline{C}_{(3)} = C_{(3)} + C_{(1,1)} + \frac{1}{12}C_{(1)} + \frac{7}{2880}C_{()}, \qquad \overline{C}_{(4)} = C_{(4)} + \text{l.o.t}$$
- GW/Hurwitz correspondence for non-singular curve X: descendents of the class of a point ω are equivalent to completed cycles

  $$\tau_k(\omega) = \frac{\overline{C}_{(k+1)}}{k!}.$$

Proof:

- Proof via topological recursion and DOSS equivalence ('19, see Generalised Zvonkine conjecture)
- Proof via localisation on the moduli space of stable maps by Leigh ('20).


ELSV
ORBIFOLD ELSV
SPIN ELSV
SPIN ORBIFOLD ELSV
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$

Conjecture (Kramer, L., Popolitov, Shadrin (2017). Now theorem.)

Let g, n be non-negative integers such that $2g-2+n>0$. Let q, r be positive integers. For a partition μ of length n and size d divisible by q, we have :

$$h^{\circ,q,r-spin}_{g;\mu} = c^{q,r}_{g,n} \cdot \prod_{i=1}^{n} \frac{(\frac{\mu_i}{qr})^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}_{g,n}(rq, q; qr - \langle \mu_i \rangle)}{\prod_{i=1}^{n}(1-\frac{\mu_i}{qr}\psi_i)}$$

where:

- $h^{\circ,q,r-spin}_{g;\mu} = [C_{id}]^{\circ}.C_{\mu} \frac{(\overline{C}_{r+1})^{b}}{b!} C_{(q,q,\ldots,q)}$
- $\mu_i = qr[\mu_i] + \langle \mu_i \rangle$, and by Riemann Hurwitz $b = (2g-2+\ell(\mu)+|\mu|/q)/r$
- $c^{q,r}_{g,n}$ is the product of possibly fractional powers of r and q depending on g, n, μ.

The proof that uses topological recursion goes through a series of papers:

- DOSS equivalence for $q=1$ (Shadrin, Spitz, Zvonkine, '13)
- DOSS equivalence for Chiodo classes (L., Popolitov, Shadrin, Zvonkine, '15)
- Quasi-polynomiality via Fock space for $r=1$ (Dunin-Barkowski, L., Popolitov, Shadrin, '15)
- Quasi-polynomiality via Fock space and conjecture (Kramer, L., Popolitov, Shadrin, '17)
- Proved in genus zero, proved for $r=2$ in any genera, via loop equation techniques (Borot, Kramer, L., Popolitov, Shadrin, '17)
- Proved in all cases extending loop equation techniques (Dunin-Barkowski, Kramer, Popolitov, Shadrin, '19)

Conjecture (Kramer, L., Popolitov, Shadrin (2017). Now theorem.)

Let g, n be non-negative integers such that $2g-2+n>0$. Let q, r be positive integers. For a partition μ of length n and size d divisible by q, we have :

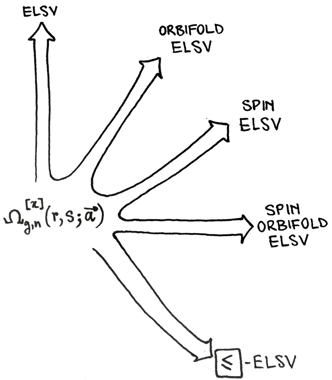
$$h^{\circ, q, r-spin}_{g;\mu} = c^{q,r}_{g,n} \cdot \prod_{i=1}^{n} \frac{(\frac{\mu_i}{qr})^{[\mu_i]}}{[\mu_i]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}_{g,n}(rq, q; qr - \langle\mu_i\rangle)}{\prod_{i=1}^{n}(1-\frac{\mu_i}{qr}\psi_i)}$$

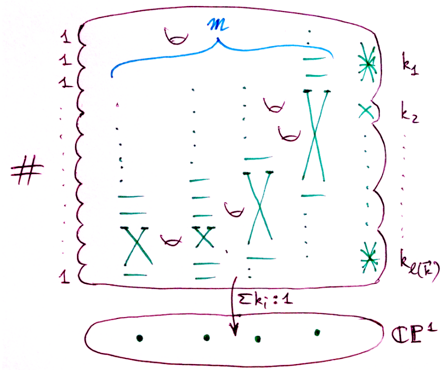
where:

- $h^{\circ, q, r-spin}_{g;\mu} = [C_{id}]^{\circ} . C_{\mu} \frac{(\overline{C}_{r+1})^b}{b!} C_{(q,q,\ldots,q)}$
- $\mu_i = qr[\mu_i] + \langle\mu_i\rangle$, and by Riemann Hurwitz $b = (2g-2+\ell(\mu)+|\mu|/q)/r$
- $c^{q,r}_{g,n}$ is the product of possibly fractional powers of r and q depending on g, n, μ.

The proof that uses topological recursion goes through a series of papers:

- DOSS equivalence for $q=1$ (Shadrin, Spitz, Zvonkine, '13)
- DOSS equivalence for Chiodo classes (L., Popolitov, Shadrin, Zvonkine, '15)
- Quasi-polynomiality via Fock space for $r=1$ (Dunin-Barkowski, L., Popolitov, Shadrin, '15)
- Quasi-polynomiality via Fock space and conjecture (Kramer, L., Popolitov, Shadrin, '17)
- Proved in genus zero, proved for $r=2$ in any genera, via loop equation techniques (Borot, Kramer, L., Popolitov, Shadrin, '17)
- Proved in all cases extending loop equation techniques (Dunin-Barkowski, Kramer, Popolitov, Shadrin, '19)


ELSV
ORBIFOLD ELSV
SPIN ELSV
SPIN ORBIFOLD ELSV
-ELSV
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$



Monotone Hurwitz numbers: labelling the cover sheets and the simple ramification points, the highest label is monotonically increasing.

Theorem (ELSV for monotone Hurwitz numbers. Alexandrov, L., Shadrin, '15)

Let g, n be non-negative integers such that $2g-2+n>0$. For a partition μ of length n and size d we have :

$$h^{\leq,\circ}_{g;\mu} = \prod_{i=1}^{n} \binom{2\mu_i}{\mu_i} \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{m=1} A_m \kappa_m \right) \prod_{i=1}^{n} \sum_{d_i=0} \frac{(2(\mu_i+d_i)-1)!!}{(2\mu_i-1)!!}$$

where:

- $h^{\leq,\circ}_{g;\mu} = [C_{id}]^{\circ}.C_{\mu} h_m(J_2,\ldots,J_d) C_{(2,2,\ldots,2)}$
- h_k is the complete homogeneous polynomial and $J_t := (1\ \ t) + (2\ \ t) + \cdots + (t-1\ \ t) \in \mathbb{Q}[\mathfrak{S}_d]$ is the t-th Jucys-Murphy element.
- By Riemann Hurwitz $m = 2g-2+n+d$.
- $\sum_{i=0}(2k+1)!!x^i = \exp(-\sum_{m=1} A_m x^m)$

Remarks:

- These numbers arise as coefficient of the HCIZ matrix model for Coulomb gas (Goulden, Guay-Paquet, Novak, '11)
- These numbers belong to ENT via $\Sigma = \mathbb{CP}^1$, $x = (z-1)/z^2$, $y=-z$, $\omega_{0,2} = \frac{dz_1 d_2}{(z_1-z_2)^2}$ (Do, Dyer, Mathews, '14)

Theorem (ELSV for monotone Hurwitz numbers. Alexandrov, L., Shadrin, '15)

Let g, n be non-negative integers such that $2g-2+n>0$. For a partition μ of length n and size d we have :

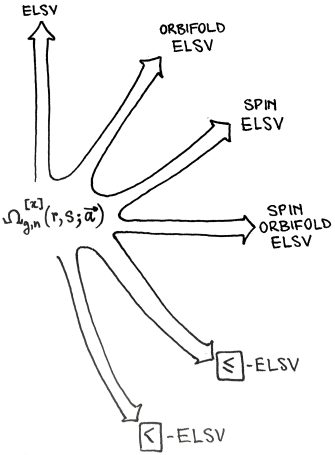
$$h^{\leq,\circ}_{g;\mu} = \prod_{i=1}^{n} \binom{2\mu_i}{\mu_i} \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{m=1} A_m \kappa_m\right) \prod_{i=1}^{n} \sum_{d_i=0} \frac{(2(\mu_i+d_i)-1)!!}{(2\mu_i-1)!!}$$

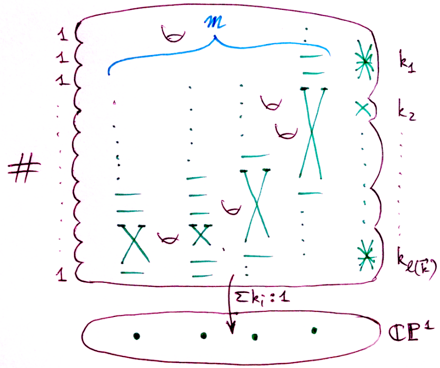
where:

- $h^{\leq,\circ}_{g;\mu} = [C_{id}]^{\circ}.C_\mu h_m(J_2,\dots,J_d)C_{(2,2,\dots,2)}$
- h_k is the complete homogeneous polynomial and $J_t := (1\ t) + (2\ t) + \cdots + (t-1\ t) \in \mathbb{Q}[\mathfrak{S}_d]$ is the t-th Jucys-Murphy element.
- By Riemann Hurwitz $m = 2g-2+n+d$.
- $\sum_{i=0}(2k+1)!!x^i = \exp(-\sum_{m=1} A_m x^m)$

Remarks:

- These numbers arise as coefficient of the HCIZ matrix model for Coulomb gas (Goulden, Guay-Paquet, Novak, '11)
- These numbers belong to ENT via $\Sigma = \mathbb{CP}^1$, $x = (z-1)/z^2$, $y = -z$, $\omega_{0,2} = \frac{dz_1 d_2}{(z_1-z_2)^2}$ (Do, Dyer, Mathews, '14)


ELSV
ORBIFOLD ELSV
SPIN ELSV
SPIN ORBIFOLD ELSV
$\Omega_{g,n}^{[x]}(r,s;\vec{a})$
≤ -ELSV
< -ELSV



Strictly monotone Hurwitz numbers or Grothendieck dessins d'enfant: labelling the cover sheets and the simple ramification points, the highest label is strictly monotonically increasing.

Theorem (ELSV for Grothendieck dessins d'enfant. Borot, Garcia-Failde, '17)

Let g, n be non-negative integers such that $2g-2+n>0$. For a partition μ of length n and size d we have :

$$h^{<,2,\circ}_{g;2\mu} = 2^g \prod_{i=1}^{n} \binom{2\mu_i}{\mu_i}\mu_i \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}(1,-1;\vec{1})\Omega^{[1/2]}(1,1;\vec{1})\Omega^{[1]}(1,1;\vec{1})}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} \sum_{h=0} \frac{[\Delta_h]}{2^{3h}(2h)!}$$

where:

- $h^{<,2,\circ}_{g;2\mu} = |Aut(\mu)|[C_{id}]^\circ . C_{(2\mu_1,\dots,2\mu_n)} \left(\sum_{\alpha:|\alpha|=d,\ell(\alpha)=b} C_\alpha\right) C_{(2,2,\dots,2)}$
- By Riemann Hurwitz $b = d/2-(2g-2+n)$
- $[\Delta_h]$ is the Poincaré dual of the boundary strata $\overline{\mathcal{M}}_{g-h,n+2h} \subset \overline{\mathcal{M}}_{g,n}$ obtained by gluing the last $2h$ marked points pairwise.

- These numbers belong to ENT via $\Sigma=\mathbb{CP}^1$, $x=z+1/z$, $y=-z$, $\omega_{0,2}=\frac{dz_1 dz_2}{(z_1-z_2)^2}$
- They are strictly monotone Hurwitz numbers via Jucys-Murphy correspondence:

$$\sigma_k(J_2,\dots,J_d) = \sum_{\alpha:|\alpha|=d,\ell(\alpha)=d-k} C_\alpha$$

where the Jucys-Murphy elements are $J_t := (1\ t)+(2\ t)+\cdots+(t-1\ t) \in \mathbb{Q}[\mathfrak{S}_d]$, and σ_k is the k-th elementary symmetric polynomial.

Theorem (ELSV for Grothendieck dessins d'enfant. Borot, Garcia-Failde, '17)

Let g, n be non-negative integers such that $2g-2+n>0$. For a partition μ of length n and size d we have :

$$h^{<,2,\circ}_{g;2\mu} = 2^g \prod_{i=1}^{n} \binom{2\mu_i}{\mu_i} \mu_i \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[1]}(1,-1;\vec{1})\Omega^{[1/2]}(1,1;\vec{1})\Omega^{[1]}(1,1;\vec{1})}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} \sum_{h=0} \frac{[\Delta_h]}{2^{3h}(2h)!}$$

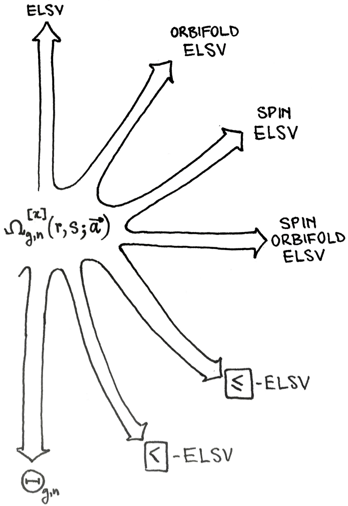
where:

- $h^{<,2,\circ}_{g;2\mu} = |Aut(\mu)|[C_{id}]^\circ . C_{(2\mu_1,\dots,2\mu_n)} \left(\sum_{\alpha:|\alpha|=d,\ell(\alpha)=b} C_\alpha\right) C_{(2,2,\dots,2)}$
- By Riemann Hurwitz $b = d/2 - (2g-2+n)$
- $[\Delta_h]$ is the Poincaré dual of the boundary strata $\overline{\mathcal{M}}_{g-h,n+2h} \subset \overline{\mathcal{M}}_{g,n}$ obtained by gluing the last $2h$ marked points pairwise.

- These numbers belong to ENT via $\Sigma = \mathbb{CP}^1$, $x = z + 1/z$, $y = -z$, $\omega_{0,2} = \frac{dz_1 d_2}{(z_1-z_2)^2}$
- They are strictly monotone Hurwitz numbers via Jucys-Murphy correspondence:

$$\sigma_k(J_2,\dots,J_d) = \sum_{\alpha:|\alpha|=d,\ell(\alpha)=d-k} C_\alpha$$

where the Jucys-Murphy elements are $J_t := (1\ \ t) + (2\ \ t) + \cdots + (t-1\ \ t) \in \mathbb{Q}[\mathfrak{S}_d]$, and σ_k is the k-th elementary symmetric polynomial.


ELSV
ORBIFOLD ELSV
SPIN ELSV
SPIN ORBIFOLD ELSV
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$
$\leq$-ELSV
$<$-ELSV
$\Theta_{g,n}$

Theorem (Norbury, 2017)

i). The following properties identify uniquely the intersection numbers of $\{\Theta_{g,n}\}_{2g-2+n>0}$:

- $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$ is of pure degree.
- $\phi_{\mathrm{irr}}^* \Theta_{g,n} = \Theta_{g-1,n+2}$ and $\phi_{h,I}^* \Theta_{g,n} = \pi_1^* \Theta_{h,|I|+1} \cdot \pi_2^* \Theta_{g-h,n-|I|+1}$ (attaching maps)
- $\pi^* \Theta_{g,n} = \psi_{n+1} \cdot \Theta_{g,n+1}$. (forgetful map)
- $\Theta_{1,1} = 3\psi_1$ (initial data)

Moreover, any such collection is of pure degree $2g-2+n$, is invariant under the $\mathfrak{S}_n$ action permutating the labels and vanishes in genus zero for any n.

ii). Such classes can be constructed as follows:

$$\Theta_{g,n} := 2^{g-1+n} \cdot [\deg = 2g-2+n].\left(\Omega^{[-1]}_{g,n}(2,-1;\vec{1})\right).$$

Moreover, $[\deg > 2g-2+n].\Theta_{g,n} = 0$ (top Chern of an actual bundle).

iii). The partition function of its intersection theory $Z^{\Theta}(\hbar,\vec{t}) = \exp\left(\sum_{g,n,\vec{d}} \frac{\hbar^{g-1}}{n!}\left\langle \Theta_{g,n} \prod \psi_i^{d_i} \prod t_{d_i}\right\rangle\right)$ is a tau function of the **KdV hierarchy**.

- $\Theta_{g,n}$ is the CohFT corresponding to the Brézin-Gross-Witten matrix model (key of the proof above)
- (Do, Norbury (2016)) ENT via the irregular Bessel spectral curve: $x(z) = z^2/2$, $y(z) = 1/z$, Virasoro, cut and join, quantum curve.
- (Witten, Stanford (2019)) $\Theta_{g,n}$ is involved in JT Gravity and Mirzakhani's recursion for volumes of moduli spaces of super Riemann surfaces.
- (Norbury (2020)) The volumes of the moduli space of super hyperbolic surfaces with geodesic boundary lengths L_i, $\langle \Theta_{g,n} \exp(2\pi^2\kappa_1 + \frac{1}{2}\sum L_i\psi_i)\rangle$, belong to ENT via $x = z^2/2$, $y = \cos(2\pi z)/z$.

Theorem (Norbury, 2017)

i). The following properties identify uniquely the intersection numbers of $\{\Theta_{g,n}\}_{2g-2+n>0}$:

- $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$ is of pure degree.
- $\phi_{\mathrm{irr}}^* \Theta_{g,n} = \Theta_{g-1,n+2}$ and $\phi_{h,I}^* \Theta_{g,n} = \pi_1^* \Theta_{h,|I|+1} \cdot \pi_2^* \Theta_{g-h,n-|I|+1}$ (attaching maps)
- $\pi^* \Theta_{g,n} = \psi_{n+1} \cdot \Theta_{g,n+1}$. (forgetful map)
- $\Theta_{1,1} = 3\psi_1$ (initial data)

Moreover, any such collection is of pure degree $2g-2+n$, is invariant under the $\mathfrak{S}_n$ action permutating the labels and vanishes in genus zero for any n.

ii). Such classes can be constructed as follows:

$$\Theta_{g,n} := 2^{g-1+n} \cdot [\deg = 2g-2+n].\left(\Omega_{g,n}^{[-1]}(2,-1;\vec{1})\right).$$

Moreover, $[\deg > 2g-2+n].\Theta_{g,n} = 0$ (top Chern of an actual bundle).

iii). The partition function of its intersection theory $Z^{\Theta}(\hbar, \vec{t}) = \exp\left(\sum_{g,n,\vec{d}} \frac{\hbar^{g-1}}{n!}\left\langle \Theta_{g,n} \prod \psi_i^{d_i} \prod t_{d_i} \right\rangle\right)$ is a tau function of the **KdV hierarchy**.

- $\Theta_{g,n}$ is the CohFT corresponding to the Brézin-Gross-Witten matrix model (key of the proof above)
- (Do, Norbury (2016)) ENT via the irregular Bessel spectral curve: $x(z) = z^2/2$, $y(z) = 1/z$, Virasoro, cut and join, quantum curve.
- (Witten, Stanford (2019)) $\Theta_{g,n}$ is involved in JT Gravity and Mirzakhani's recursion for volumes of moduli spaces of super Riemann surfaces.
- (Norbury (2020)) The volumes of the moduli space of super hyperbolic surfaces with geodesic boundary lengths L_i, $\langle \Theta_{g,n} \exp(2\pi^2 \kappa_1 + \frac{1}{2}\sum L_i \psi_i)\rangle$, belong to ENT via $x = z^2/2$, $y = \cos(2\pi z)/z$.

Theorem (Norbury, 2017)

i). The following properties identify uniquely the intersection numbers of $\{\Theta_{g,n}\}_{2g-2+n>0}$:

- $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$ is of pure degree.
- $\phi_{\text{irr}}^* \Theta_{g,n} = \Theta_{g-1,n+2}$ and $\phi_{h,I}^* \Theta_{g,n} = \pi_1^* \Theta_{h,|I|+1} \cdot \pi_2^* \Theta_{g-h,n-|I|+1}$ (attaching maps)
- $\pi^* \Theta_{g,n} = \psi_{n+1} \cdot \Theta_{g,n+1}$. (forgetful map)
- $\Theta_{1,1} = 3\psi_1$ (initial data)

Moreover, any such collection is of pure degree $2g-2+n$, is invariant under the $\mathfrak{S}_n$ action permutating the labels and vanishes in genus zero for any n.

ii). Such classes can be constructed as follows:

$$\Theta_{g,n} := 2^{g-1+n} \cdot [\deg = 2g-2+n].\left(\Omega_{g,n}^{[-1]}(2,-1;\vec{1})\right).$$

Moreover, $[\deg > 2g-2+n].\Theta_{g,n} = 0$ (top Chern of an actual bundle).

iii). The partition function of its intersection theory
$Z^{\Theta}(\hbar,\vec{t}) = \exp\left(\sum_{g,n,\vec{d}} \frac{\hbar^{g-1}}{n!}\left\langle \Theta_{g,n} \prod \psi_i^{d_i} \prod t_{d_i}\right\rangle\right)$ is a tau function of the **KdV hierarchy**.

- $\Theta_{g,n}$ is the CohFT corresponding to the Brézin-Gross-Witten matrix model (key of the proof above)
- (Do, Norbury (2016)) ENT via the irregular Bessel spectral curve: $x(z) = z^2/2$, $y(z) = 1/z$, Virasoro, cut and join, quantum curve.
- (Witten, Stanford (2019)) $\Theta_{g,n}$ is involved in JT Gravity and Mirzakhani's recursion for volumes of moduli spaces of super Riemann surfaces.
- (Norbury (2020)) The volumes of the moduli space of super hyperbolic surfaces with geodesic boundary lengths L_i, $\langle \Theta_{g,n} \exp(2\pi^2\kappa_1 + \frac{1}{2}\sum L_i\psi_i)\rangle$, belong to ENT via $x = z^2/2$, $y = \cos(2\pi z)/z$.

Theorem (Norbury, 2017)

i). The following properties identify uniquely the intersection numbers of $\{\Theta_{g,n}\}_{2g-2+n>0}$:

- $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$ is of pure degree.
- $\phi_{\text{irr}}^*\Theta_{g,n} = \Theta_{g-1,n+2}$ and $\phi_{h,I}^*\Theta_{g,n} = \pi_1^*\Theta_{h,|I|+1} \cdot \pi_2^*\Theta_{g-h,n-|I|+1}$ (attaching maps)
- $\pi^*\Theta_{g,n} = \psi_{n+1} \cdot \Theta_{g,n+1}$. (forgetful map)
- $\Theta_{1,1} = 3\psi_1$ (initial data)

Moreover, any such collection is of pure degree $2g-2+n$, is invariant under the $\mathfrak{S}_n$ action permutating the labels and vanishes in genus zero for any n.

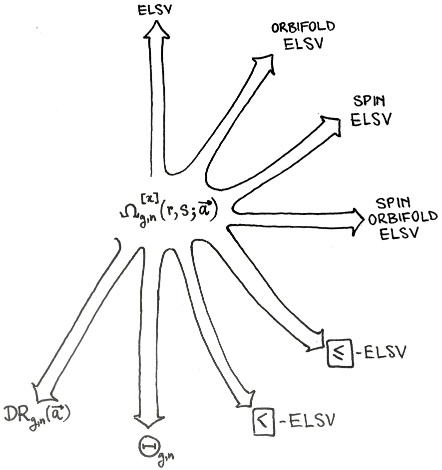
ii). Such classes can be constructed as follows:

$$\Theta_{g,n} := 2^{g-1+n} \cdot [\deg = 2g-2+n].\left(\Omega_{g,n}^{[-1]}(2,-1;\vec{1})\right).$$

Moreover, $[\deg > 2g-2+n].\Theta_{g,n} = 0$ (top Chern of an actual bundle).

iii). The partition function of its intersection theory $Z^{\Theta}(\hbar,\vec{t}) = \exp\left(\sum_{g,n,\vec{d}} \frac{\hbar^{g-1}}{n!}\left\langle \Theta_{g,n}\prod \psi_i^{d_i} \prod t_{d_i}\right\rangle\right)$ is a tau function of the **KdV hierarchy**.

- $\Theta_{g,n}$ is the CohFT corresponding to the Brézin-Gross-Witten matrix model (key of the proof above)
- (Do, Norbury (2016)) ENT via the irregular Bessel spectral curve: $x(z) = z^2/2$, $y(z) = 1/z$, Virasoro, cut and join, quantum curve.
- (Witten, Stanford (2019)) $\Theta_{g,n}$ is involved in JT Gravity and Mirzakhani's recursion for volumes of moduli spaces of super Riemann surfaces.
- (Norbury (2020)) The volumes of the moduli space of super hyperbolic surfaces with geodesic boundary lengths L_i, $\langle \Theta_{g,n} \exp(2\pi^2\kappa_1 + \frac{1}{2}\sum L_i\psi_i)\rangle$, belong to ENT via $x = z^2/2$, $y = \cos(2\pi z)/z$.


ELSV
ORBIFOLD ELSV
SPIN ELSV
SPIN ORBIFOLD ELSV
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$
$\leq$ -ELSV
$<$ -ELSV
$\Theta_{g,n}$
$DR_{g,n}(\vec{a})$

Definition (Double ramification cycle)

Let A be a zero sum vector of integers of length n, $a = a_+ \sqcup a_0 \sqcup a_-$. Let $\mathrm{stab} : \overline{\mathcal{M}}_{g,a_0}(\mathbb{P}^1, a_-(0), a_+(\infty))^\sim \longrightarrow \overline{\mathcal{M}}_{g,n}$ the map stabilising the target of the moduli space of stable maps to rubber $\mathbb{P}^1$ relative to the partitions over zero and infinity defined by the positive and the negative elements of A. Then

$$DR_{g,n}(a_1, \dots, a_n) := \mathrm{stab}_*[\overline{\mathcal{M}}_{g,a_0}(\mathbb{P}^1, a_-(0), a_+(\infty))^\sim]^{\mathrm{vir}}$$

Why Double Ramification cycles?

- Eliashberg problem: what is a good compactification?
- Hain formula on compact type, polynomial in a_i of degree $2g$
- Faber and Pandharipande prove it tautological of degree g
- Central object in the construction of Buryak integrable hierarchy from a Cohft, conjecturally Miura equivalent to and therefore extending Dubrovin Zhang construction.
- Many different approaches in algebraic geometry for its description/construction...

Theorem ((Janda, Pixton, Pandharipande, Zvonkine), Proof of Pixton conjecture)

$$DR_{g,n}(a_1, \dots, a_n) = \left(r \cdot [\deg = g].\Omega^{[1]}_{g,n}(r, r, a_1, \dots, a_n)\right)\Big|_{r=0} \qquad r >> a_i$$

Definition (Double ramification cycle)

Let A be a zero sum vector of integers of length n, $a = a_+ \sqcup a_0 \sqcup a_-$. Let $\mathrm{stab} : \overline{\mathcal{M}}_{g,a_0}(\mathbb{P}^1, a_-(0), a_+(\infty))^{\sim} \longrightarrow \overline{\mathcal{M}}_{g,n}$ the map stabilising the target of the moduli space of stable maps to rubber $\mathbb{P}^1$ relative to the partitions over zero and infinity defined by the positive and the negative elements of A. Then

$$DR_{g,n}(a_1, \ldots, a_n) := \mathrm{stab}_*[\overline{\mathcal{M}}_{g,a_0}(\mathbb{P}^1, a_-(0), a_+(\infty))^{\sim}]^{\mathrm{vir}}$$

Why Double Ramification cycles?

- Eliashberg problem: what is a good compactification?
- Hain formula on compact type, polynomial in a_i of degree $2g$
- Faber and Pandharipande prove it tautological of degree g
- Central object in the construction of Buryak integrable hierarchy from a Cohft, conjecturally Miura equivalent to and therefore extending Dubrovin Zhang construction.
- Many different approaches in algebraic geometry for its description/construction...

Theorem ((Janda, Pixton, Pandharipande, Zvonkine), Proof of Pixton conjecture)

$$DR_{g,n}(a_1, \ldots, a_n) = \left(r \cdot [\deg = g].\Omega^{[1]}_{g,n}(r, r, a_1, \ldots, a_n) \right) \Big|_{r=0} \qquad r >> a_i$$

Definition (Double ramification cycle)

Let A be a zero sum vector of integers of length n, $a = a_+ \sqcup a_0 \sqcup a_-$. Let $\mathrm{stab} : \overline{\mathcal{M}}_{g,a_0}(\mathbb{P}^1, a_-(0), a_+(\infty))^\sim \longrightarrow \overline{\mathcal{M}}_{g,n}$ the map stabilising the target of the moduli space of stable maps to rubber $\mathbb{P}^1$ relative to the partitions over zero and infinity defined by the positive and the negative elements of A. Then

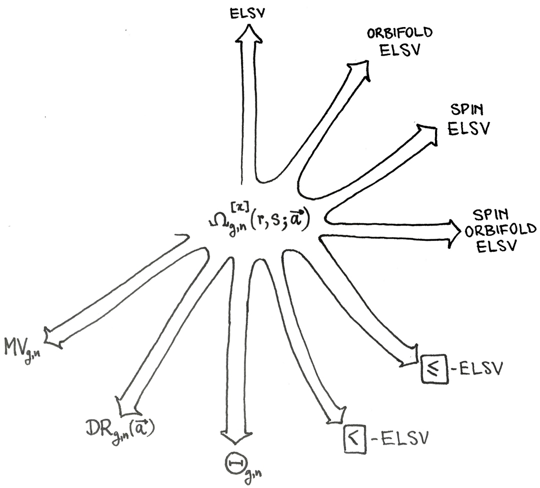
$$DR_{g,n}(a_1, \ldots, a_n) := \mathrm{stab}_*[\overline{\mathcal{M}}_{g,a_0}(\mathbb{P}^1, a_-(0), a_+(\infty))^\sim]^{\mathrm{vir}}$$

Why Double Ramification cycles?

- Eliashberg problem: what is a good compactification?
- Hain formula on compact type, polynomial in a_i of degree $2g$
- Faber and Pandharipande prove it tautological of degree g
- Central object in the construction of Buryak integrable hierarchy from a Cohft, conjecturally Miura equivalent to and therefore extending Dubrovin Zhang construction.
- Many different approaches in algebraic geometry for its description/construction...

Theorem ((Janda, Pixton, Pandharipande, Zvonkine), Proof of Pixton conjecture)

$$DR_{g,n}(a_1, \ldots, a_n) = \left(r \cdot [\deg = g].\Omega^{[1]}_{g,n}(r, r, a_1, \ldots, a_n) \right) \Big|_{r=0} \qquad r >> a_i$$


ELSV
ORBIFOLD ELSV
SPIN ELSV
SPIN ORBIFOLD ELSV
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$
$MV_{g,n}$
$DR_{g,n}(\vec{a})$
$\Theta_{g,n}$
-ELSV
-ELSV

Definition (Masur-Veech polynomials)

For $2g-2+n>0$ define $V\Omega^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n)$ as

$$\sum_{\Gamma\in\mathbf{G}_{g,n}}\frac{1}{|\mathrm{Aut}\,\Gamma|}\int_{\mathbb{R}_+^{E_\Gamma}}\prod_{v\in V_\Gamma}V\Omega^{\mathrm{K}}_{h(v),k(v)}\big((\ell_e)_{e\in E(v)},(L_\lambda)_{\lambda\in\Lambda(v)}\big)\prod_{e\in E_\Gamma}\frac{\ell_e\mathrm{d}\ell_e}{e^{\ell_e}-1},$$

which is of total degree $3g-3+n$, where $V\Omega^{\mathrm{K}}_{g,n}(L_1,\ldots,L_n)=\Big\langle\exp\big(\tfrac{1}{2}\sum_{i=1}^{n}L_i^2\psi_i\big)\Big\rangle$

Theorem (Delecroix, Goujard, Zograf, Zorich)

Let $MV_{g,n}$ be the Masur-Veech volumes associated with the principal strata of the moduli spaces of quadratic differentials of genus g with n marked points. Then

$$MV_{g,n}=\frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!}\,V\Omega^{\mathrm{MV}}_{g,n}(0,\ldots,0). \qquad (1)$$

Definition (Masur-Veech polynomials)

For $2g-2+n>0$ define $V\Omega^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n)$ as

$$\sum_{\Gamma\in\mathbf{G}_{g,n}}\frac{1}{|\mathrm{Aut}\,\Gamma|}\int_{\mathbb{R}_+^{E_\Gamma}}\prod_{v\in V_\Gamma}V\Omega^{\mathrm{K}}_{h(v),k(v)}\big((\ell_e)_{e\in E(v)},(L_\lambda)_{\lambda\in\Lambda(v)}\big)\prod_{e\in E_\Gamma}\frac{\ell_e\mathrm{d}\ell_e}{e^{\ell_e}-1},$$

which is of total degree $3g-3+n$, where $V\Omega^{\mathrm{K}}_{g,n}(L_1,\ldots,L_n)=\Big\langle\exp\big(\frac{1}{2}\sum_{i=1}^n L_i^2\psi_i\big)\Big\rangle$

Theorem (Delecroix, Goujard, Zograf, Zorich)

Let $MV_{g,n}$ be the Masur-Veech volumes associated with the principal strata of the moduli spaces of quadratic differentials of genus g with n marked points. Then

$$MV_{g,n}=\frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!}\,V\Omega^{\mathrm{MV}}_{g,n}(0,\ldots,0). \tag{1}$$

Theorem (Topological Recursion for Masur-Veech volumes. Andersen, Borot, Charbonnier, Delecroix, Giacchetto, L , Wheeler 2019)

Masur-Veech volumes of quadratic differentials belong to ENT. More precisely, the spectral curve given on the Riemann sphere by

$$x(z) = \frac{z^2}{2}, \qquad y(z) = -z, \qquad \omega_{0,2}^{\mathrm{MV}}(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2} + \frac{1}{2} \sum_{m \in \mathbb{Z}^*} \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2 + m)^2},$$

produces TR output expanded in the Riemann-Hurwitz functions $\zeta_{\mathrm{H}}(\ell, z) = \sum_{m \in \mathbb{Z}} (z+m)^{-\ell}$

$$\omega_{g,n}^{\mathrm{MV}}(z_1, \ldots, z_n) = \sum_{d_1 + \cdots + d_n \leq 3g-3+n} F_{g,n}^{\mathrm{MV}}[d_1, \ldots, d_n] \bigotimes_{i=1}^{n} \zeta_{\mathrm{H}}(2d_i + 2; z_i)\, \mathrm{d}z_i.$$

where $F_{g,n}^{\mathrm{MV}}$ are the coefficients of $V\Omega_{g,n}^{\mathrm{MV}}$ in the expansion:

$$V\Omega_{g,n}^{\mathrm{MV}}(L_1, \ldots, L_n) = \sum_{d_1 + \cdots + d_n \leq 3g-3+n} F_{g,n}^{\mathrm{MV}}[d_1, \ldots, d_n] \prod_{i=1}^{n} \frac{L_i^{2d_i}}{(2d_i+1)!}.$$

- First compute the entire collection of polynomials $V\Omega_{g,n}^{\mathrm{MV}}$ up to a certain level in the Euler characteristic, then extract MV volumes.

Conjecture (Masur-Veech volumes for fixed genus)

For any fixed genus $g \geq 0$, there exist polynomials $p_g, q_g \in \mathbb{Q}[n]$ such that, for any $n \geq 0$

$$\frac{MV_{g,n}}{\pi^{6g-6+2n}} = 2^n \frac{(2g-3+n)!(4g-4+n)!}{(6g-7+2n)!} \left(p_g(n) + \gamma_{2g-3+n}\, q_g(n)\right), \qquad \gamma_k = \frac{1}{4^k}\binom{2k}{k}.$$

of explicit degrees growing linearly with the genus.

Theorem (Chen, Möller, Sauvaget)

MV conjecture for fixed g holds true. Key ingredient for the proof: ELSV for MV.

$$\frac{MV_{g,n}}{\pi^{6g-6+2n}} = 2^{2g+1} \frac{(-1)^{3g-3+n}(4g-4+n)!}{(6g-7+2n)!} \int_{\overline{\mathcal{M}}_{g,n}} \left(\Omega^{[1]}(1,-1;\vec{1})\right)^{-1}$$

Moreover, factorisation Chiodo classes allows expression in Hodge integrals against ψ_i^2.

Corollary (Borot, Giacchetto, L. Appendix to result above)

$(\Omega^{[1]}(1,-1;\vec{1}))^{-1} = \Omega^{[-1]}(1,2;\vec{1})$, hence $MV_{g,n}$ belong to ENT in a second inequivalent way, via

$$\Sigma = \mathbb{CP}^1, \qquad x(z) = -z - \log(z), \qquad y(z) = z^2, \qquad \omega_{0,2}(z_1,z_2) = \frac{dz_1 \otimes dz_2}{(z_1-z_2)^2}$$

- Kazarian: KP integrability of Hodge integrals to compute MV volumes recursively
- Yang, Zagier, Zhang: ILW integrability to compute MV volumes recursively

Conjecture (Masur-Veech volumes for fixed genus)

For any fixed genus $g \geq 0$, there exist polynomials $p_g, q_g \in \mathbb{Q}[n]$ such that, for any $n \geq 0$

$$\frac{MV_{g,n}}{\pi^{6g-6+2n}} = 2^n \frac{(2g-3+n)!(4g-4+n)!}{(6g-7+2n)!} \big(p_g(n) + \gamma_{2g-3+n}\, q_g(n) \big), \qquad \gamma_k = \frac{1}{4^k}\binom{2k}{k}.$$

of explicit degrees growing linearly with the genus.

Theorem (Chen, Möller, Sauvaget)

MV conjecture for fixed g holds true. Key ingredient for the proof: ELSV for MV.

$$\frac{MV_{g,n}}{\pi^{6g-6+2n}} = 2^{2g+1} \frac{(-1)^{3g-3+n}(4g-4+n)!}{(6g-7+2n)!} \int_{\overline{\mathcal{M}}_{g,n}} \Big(\Omega^{[1]}(1,-1;\vec{1}) \Big)^{-1}$$

Moreover, factorisation Chiodo classes allows expression in Hodge integrals against ψ_i^2.

Corollary (Borot, Giacchetto, L. Appendix to result above)

$(\Omega^{[1]}(1,-1;\vec{1}))^{-1} = \Omega^{[-1]}(1,2;\vec{1})$, hence $MV_{g,n}$ belong to ENT in a second inequivalent way, via

$$\Sigma = \mathbb{CP}^1, \qquad x(z) = -z - \log(z), \qquad y(z) = z^2, \qquad \omega_{0,2}(z_1,z_2) = \frac{dz_1 \otimes dz_2}{(z_1-z_2)^2}$$

- Kazarian: KP integrability of Hodge integrals to compute MV volumes recursively
- Yang, Zagier, Zhang: ILW integrability to compute MV volumes recursively

Conjecture (Masur-Veech volumes for fixed genus)

For any fixed genus $g \geq 0$, there exist polynomials $p_g, q_g \in \mathbb{Q}[n]$ such that, for any $n \geq 0$

$$\frac{MV_{g,n}}{\pi^{6g-6+2n}} = 2^n \frac{(2g-3+n)!(4g-4+n)!}{(6g-7+2n)!} \big(p_g(n) + \gamma_{2g-3+n}\, q_g(n)\big), \qquad \gamma_k = \frac{1}{4^k}\binom{2k}{k}.$$

of explicit degrees growing linearly with the genus.

Theorem (Chen, Möller, Sauvaget)

MV conjecture for fixed g holds true. Key ingredient for the proof: ELSV for MV.

$$\frac{MV_{g,n}}{\pi^{6g-6+2n}} = 2^{2g+1} \frac{(-1)^{3g-3+n}(4g-4+n)!}{(6g-7+2n)!} \int_{\overline{\mathcal{M}}_{g,n}} \left(\Omega^{[1]}(1,-1;\vec{1})\right)^{-1}$$

Moreover, factorisation Chiodo classes allows expression in Hodge integrals against ψ_i^2.

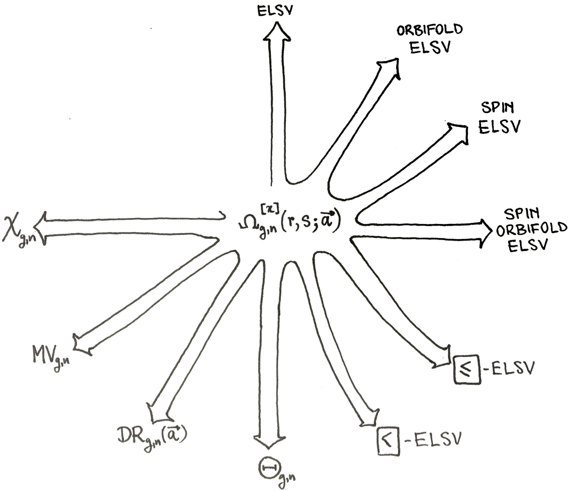
Corollary (Borot, Giacchetto, L. Appendix to result above)

$(\Omega^{[1]}(1,-1;\vec{1}))^{-1} = \Omega^{[-1]}(1,2;\vec{1})$, hence $MV_{g,n}$ belong to ENT in a second inequivalent way, via

$$\Sigma = \mathbb{CP}^1, \qquad x(z) = -z - \log(z), \qquad y(z) = z^2, \qquad \omega_{0,2}(z_1,z_2) = \frac{dz_1 \otimes dz_2}{(z_1-z_2)^2}$$

- Kazarian: KP integrability of Hodge integrals to compute MV volumes recursively
- Yang, Zagier, Zhang: ILW integrability to compute MV volumes recursively

g	$p_g(n)$	$q_g(n)$
0	0	$\frac{1}{4}$
1	$\frac{1}{6}$	$\frac{1}{6}$
2	$\frac{5}{36}$	$\frac{28}{135}n + \frac{7}{18}$
3	$\frac{245}{3888}n + \frac{643}{1944}$	$\frac{1784}{8505}n + \frac{6523}{8505}$
4	$\frac{1757}{23328}n + \frac{95413}{194400}$	$\frac{1186528}{23455575}n^2 + \frac{40882696}{54729675}n + \frac{5951381}{2296350}$
5	$\frac{38213}{3359232}n^2 + \frac{4218671}{16796160}n + \frac{63657059}{48988800}$	$\frac{83632064}{1196234325}n^2 + \frac{50144427856}{41868201375}n + \frac{63849553}{12629925}$
6	$\frac{59406613}{3325639680}n^2 + \frac{11411443987}{27713664000}n + \frac{61888029881}{26453952000}$	$\frac{2562397434368}{352859220016875}n^3 + \frac{185272285982144}{640374140030625}n^2 + \frac{9008283258227896}{2470014540118125}n + \frac{1636294928657}{110827591875}$


ELSV
ORBIFOLD ELSV
SPIN ELSV
SPIN ORBIFOLD ELSV
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$
$\chi_{g,n}$
$MV_{g,n}$
$DR_{g,n}(\vec{a})$
$\Theta_{g,n}$
⩽ -ELSV
< -ELSV

Remark (Giacchetto, L, Norbury)

$$\chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \Omega^{[1]}_{g,n}(1, -1; \vec{1})$$

- Euler characteristic as integral of the log cotangent + def Chiodo classes + Serre duality
- $MV_{g,n}$ are proportional to the integral of $\Omega^{[1]}_{g,n}(1, -1; \vec{1})^{-1}$
- Application:

$$\chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} c(\mathbb{E}^\vee) \exp\Big(- \sum_{m \geq 1} \frac{1}{m} \kappa_m \Big)$$

- New proof of Harer-Zager formula for $\chi_{g,n}$ (expand the exp in ψ + string, dilaton, initial condition for $n = 0$)
- Tested computationally via admcycles Sage package (Delecroix, Schmitt, van Zelm), new library with Schmitt with all ELSV formulae.

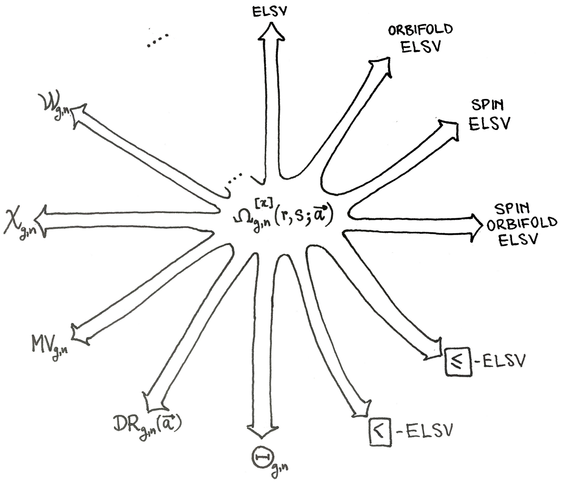
Remark (Giacchetto, L, Norbury)

$$\chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \Omega^{[1]}_{g,n}(1, -1; \vec{1})$$

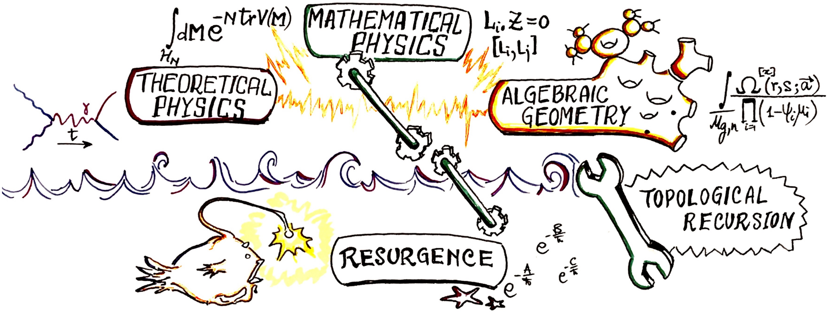
- Euler characteristic as integral of the log cotangent + def Chiodo classes + Serre duality
- $MV_{g,n}$ are proportional to the integral of $\Omega^{[1]}_{g,n}(1, -1; \vec{1})^{-1}$
- Application:

$$\chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} c(\mathbb{E}^{\vee}) \exp\Big(- \sum_{m \geq 1} \frac{1}{m} \kappa_m \Big)$$

- New proof of Harer-Zager formula for $\chi_{g,n}$ (expand the exp in ψ + string, dilaton, initial condition for $n = 0$)
- Tested computationally via admcycles Sage package (Delecroix, Schmitt, van Zelm), new library with Schmitt with all ELSV formulae.


ELSV
ORBIFOLD ELSV
SPIN ELSV
SPIN ORBIFOLD ELSV
≤ -ELSV
< -ELSV
$\Theta_{g,n}$
$DR_{g,n}(\vec{a})$
$MV_{g,n}$
$\chi_{g,n}$
$W_{g,n}$
$\Omega^{[x]}_{g,n}(r,s;\vec{a})$

What about the integrability of Chiodo integrals for arbitrary parameters?



Theorem (Kazarian, '08)

The generating series of Hodge integrals in the formal commuting variables u, T_i,

$$G(u;\vec{T}) = \sum_{j,k_j=0} (-u^2)^j \langle \lambda_j \, \tau_0^{k_0} \, \tau_1^{k_1} \ldots \rangle \frac{T_0^{k_0}}{k_0!} \frac{T_1^{k_1}}{k_1!} \ldots, \quad (2)$$

is a **solution of the KP hierarchy** with respect to the variables q_i (identically in u), after the following linear triangular change of variables $T_i = T_i(q_j)$:

$$\phi_0(u,z) := z, \quad \phi_{k+1}(u,z) := D^{k+1}\,\phi_0(u,z), \quad D := (u+z)^2 z \frac{d}{dz}$$

For instance

$$\phi_0 = z, \quad \phi_1 = u^2 z + 2uz^2 + z^3, \quad \phi_2 = u^4 z + 6u^3 z^2 + 12u^2 z^3 + 10uz^4 + 3z^5, \quad \ldots.$$

Then T_k is obtained form ϕ_k by replacing z^m by q_m in each monomial. For instance

$$T_0 = q_1, \quad T_1 = u^2\, q_1 + 2\,u\,q_2 + q_3, \quad T_2 = u^4\, q_1 + 6\,u^3\, q_2 + 12\,u^2\, q_3 + 10\,u\,q_4 + 3\,q_5$$

Equivalently, T_k is given by the following recursive equation

$$T_{k+1} = \sum_{m\geq 1} m\,(u^2\, q_m + 2\,u\,q_{m+1} + q_{m+2}) \frac{d}{dq_m} T_k, \qquad T_0 = q_1.$$

Theorem (Kazarian, '08)

The generating series of Hodge integrals in the formal commuting variables u, T_i,

$$G(u;\vec{T}) = \sum_{j,k_j=0} (-u^2)^j \langle \lambda_j \, \tau_0^{k_0} \, \tau_1^{k_1} \, \dots \rangle \frac{T_0{}^{k_0}}{k_0!} \frac{T_1{}^{k_1}}{k_1!} \dots, \tag{2}$$

is a **solution of the KP hierarchy** with respect to the variables q_i (identically in u), after the following linear triangular change of variables $T_i = T_i(q_j)$:

$$\phi_0(u,z) := z, \quad \phi_{k+1}(u,z) := D^{k+1}\,\phi_0(u,z), \quad D := (u+z)^2 z \, \frac{d}{dz}$$

For instance

$$\phi_0 = z, \quad \phi_1 = u^2 z + 2uz^2 + z^3, \quad \phi_2 = u^4 z + 6u^3 z^2 + 12u^2 z^3 + 10uz^4 + 3z^5, \quad \dots.$$

Then T_k is obtained form ϕ_k by replacing z^m by q_m in each monomial. For instance

$$T_0 = q_1, \quad T_1 = u^2\,q_1 + 2\,u\,q_2 + q_3, \quad T_2 = u^4\,q_1 + 6\,u^3\,q_2 + 12\,u^2\,q_3 + 10\,u\,q_4 + 3\,q_5$$

Equivalently, T_k is given by the following recursive equation

$$T_{k+1} = \sum_{m\geq 1} m\,(u^2\,q_m + 2\,u\,q_{m+1} + q_{m+2})\,\frac{d}{dq_m} T_k, \qquad T_0 = q_1.$$

Theorem (Kazarian, '08)

The generating series of Hodge integrals in the formal commuting variables u, T_i,

$$G(u;\vec{T}) = \sum_{j,k_j=0} (-u^2)^j \langle \lambda_j \tau_0^{k_0} \tau_1^{k_1} \dots \rangle \frac{T_0{}^{k_0}}{k_0!} \frac{T_1{}^{k_1}}{k_1!} \dots, \qquad (2)$$

is a **solution of the KP hierarchy** with respect to the variables q_i (identically in u), after the following linear triangular change of variables $T_i = T_i(q_j)$:

$$\phi_0(u,z) := z, \quad \phi_{k+1}(u,z) := D^{k+1}\,\phi_0(u,z), \quad D := (u+z)^2 z\,\frac{d}{dz}$$

For instance

$$\phi_0 = z, \quad \phi_1 = u^2z + 2uz^2 + z^3, \quad \phi_2 = u^4z + 6u^3z^2 + 12u^2z^3 + 10uz^4 + 3z^5, \quad \dots.$$

Then T_k is obtained form ϕ_k by replacing z^m by q_m in each monomial. For instance

$$T_0 = q_1, \quad T_1 = u^2\,q_1 + 2\,u\,q_2 + q_3, \quad T_2 = u^4\,q_1 + 6\,u^3\,q_2 + 12\,u^2\,q_3 + 10\,u\,q_4 + 3\,q_5$$

Equivalently, T_k is given by the following recursive equation

$$T_{k+1} = \sum_{m\geq 1} m\,(u^2\,q_m + 2\,u\,q_{m+1} + q_{m+2})\,\frac{d}{dq_m}T_k, \qquad T_0 = q_1.$$

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta; \vec{p}) = \sum_{n \geq 1} \frac{1}{n!} \sum_{g, k_1, \ldots, k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j, d_i} (-1)^j \lambda_j \prod_{i=1}^{n} \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k \geq 1} \frac{k^{k+d}}{k!} \beta^{k + \frac{1}{3} + \frac{2}{3} d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u, \vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta} x(z) = -(2z + \beta z^2)\, z \frac{d}{dz} x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing $(0, 1)$ and $(0, 2)$ unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta;\vec{p}) = \sum_{n\geq 1} \frac{1}{n!} \sum_{g,k_1,\dots,k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j,d_i} (-1)^j \lambda_j \prod_{i=1}^n \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k\geq 1} \frac{k^{k+d}}{k!} \beta^{k+\frac{1}{3}+\frac{2}{3}d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u, \vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta}x(z) = -(2z + \beta z^2) z \frac{d}{dz}x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing (0, 1) and (0, 2) unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta;\vec{p}) = \sum_{n\geq 1}\frac{1}{n!}\sum_{g,k_1,\dots,k_n} h_{g;\vec{k}}\beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^{n}\frac{k_i^{k_i}}{k_i!}\left\langle \sum_{j,d_i}(-1)^j\lambda_j\prod_{i=1}^{n}\psi_i^{d_i}\right\rangle$
- Set $T_d = \sum_{k\geq 1}\frac{k^{k+d}}{k!}\beta^{k+\frac{1}{3}+\frac{2}{3}d}p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u,\vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z}e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta}x(z) = -(2z+\beta z^2)z\frac{d}{dz}x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing (0, 1) and (0, 2) unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta; \vec{p}) = \sum_{n\geq 1} \frac{1}{n!} \sum_{g,k_1,\ldots,k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j,d_i} (-1)^j \lambda_j \prod_{i=1}^n \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k\geq 1} \frac{k^{k+d}}{k!} \beta^{k+\frac{1}{3}+\frac{2}{3}d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u, \vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta} x(z) = -(2z + \beta z^2) z \frac{d}{dz} x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing (0, 1) and (0, 2) unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta; \vec{p}) = \sum_{n\geq 1} \frac{1}{n!} \sum_{g,k_1,\dots,k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j,d_i} (-1)^j \lambda_j \prod_{i=1}^n \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k\geq 1} \frac{k^{k+d}}{k!} \beta^{k+\frac{1}{3}+\frac{2}{3}d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u, \vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta} x(z) = -(2z + \beta z^2) z \frac{d}{dz} x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing (0, 1) and (0, 2) unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta;\vec{p}) = \sum_{n\geq 1} \frac{1}{n!} \sum_{g,k_1,\dots,k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j,d_i} (-1)^j \lambda_j \prod_{i=1}^n \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k\geq 1} \frac{k^{k+d}}{k!} \beta^{k+\frac{1}{3}+\frac{2}{3}d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u, \vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta} x(z) = -(2z + \beta z^2)\, z \frac{d}{dz} x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing (0, 1) and (0, 2) unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta;\vec{p}) = \sum_{n\geq 1} \frac{1}{n!} \sum_{g,k_1,\dots,k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j,d_i} (-1)^j \lambda_j \prod_{i=1}^n \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k\geq 1} \frac{k^{k+d}}{k!} \beta^{k+\frac{1}{3}+\frac{2}{3}d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u,\vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta} x(z) = -(2z + \beta z^2)\, z \frac{d}{dz} x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing (0, 1) and (0, 2) unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta;\vec{p}) = \sum_{n\geq 1} \frac{1}{n!} \sum_{g,k_1,\dots,k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j,d_i} (-1)^j \lambda_j \prod_{i=1}^n \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k\geq 1} \frac{k^{k+d}}{k!} \beta^{k+\frac{1}{3}+\frac{2}{3}d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u,\vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta}x(z) = -(2z + \beta z^2)\, z \frac{d}{dz} x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing (0, 1) and (0, 2) unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta;\vec{p}) = \sum_{n\geq 1}\frac{1}{n!}\sum_{g,k_1,\ldots,k_n} h_{g;\vec{k}}\beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!}\left\langle \sum_{j,d_i}(-1)^j\lambda_j\prod_{i=1}^n \psi_i^{d_i}\right\rangle$
- Set $T_d = \sum_{k\geq 1}\frac{k^{k+d}}{k!}\beta^{k+\frac{1}{3}+\frac{2}{3}d}p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u,\vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z}e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta}x(z) = -(2z+\beta z^2)\,z\frac{d}{dz}x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing $(0,1)$ and $(0,2)$ unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta; \vec{p}) = \sum_{n \geq 1} \frac{1}{n!} \sum_{g, k_1, \dots, k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j, d_i} (-1)^j \lambda_j \prod_{i=1}^n \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k \geq 1} \frac{k^{k+d}}{k!} \beta^{k + \frac{1}{3} + \frac{2}{3} d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u, \vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta} x(z) = -(2z + \beta z^2) z \frac{d}{dz} x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing (0, 1) and (0, 2) unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Corollary (Kazarian)

Witten Conjecture / Kontsevich Theorem for $u = 0$.

Strategy of the proof:

- By Okounkov H is a solution of KP in the p_i: $H(\beta;\vec{p}) = \sum_{n\geq 1} \frac{1}{n!} \sum_{g,k_1,\dots,k_n} h_{g;\vec{k}} \beta^m p_{\vec{k}}$.
- Plug in ELSV formula: $h_{g;\vec{k}} = \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \left\langle \sum_{j,d_i} (-1)^j \lambda_j \prod_{i=1}^n \psi_i^{d_i} \right\rangle$
- Set $T_d = \sum_{k\geq 1} \frac{k^{k+d}}{k!} \beta^{k+\frac{1}{3}+\frac{2}{3}d} p_k$ and $u = \beta^{\frac{1}{3}}$.
- Notice $H - H_{0,1} - H_{0,2} = G(u,\vec{T})$
- Control the T change of variable: $p_k \leftrightarrow x^k$, $q_k \leftrightarrow z^k$ via $x(z) := \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$
- Express the flow along the time β as a vector field $\frac{d}{d\beta} x(z) = -(2z + \beta z^2)\, z \frac{d}{dz} x(z)$ that can be represented as an element of $\widehat{\mathfrak{gl}(\infty)}$. Then prove KP integrability for $\beta = 0$.
- Show that removing $(0,1)$ and $(0,2)$ unstable contribution does not spoil the integrability

Remarks/questions:

- Hurwitz numbers obey KP and they are in ENT. Their CohFT is Hodge.
- What is the integrability of Hodge? The change of variable given by the spectral curve preserves KP. As a result, Hodge also obeys KP.
- The spectral curve acts as a transfer of integrability between the enumerative problem and the CohFT. How general is this behaviour?

Theorem (Giacchetto, L, Norbury)

Let q, r be positive integers. The generating series Chiodo integrals in the formal commuting variables u, $\{T_{d,a}\}_{d \geq 0, a=1,\dots,qr}$,

$$G^{qr,q}(u;\vec{T}) = \sum_{g,n} \frac{1}{n!} \sum_{a_1,\dots,a_n=1}^{qr} \int_{\overline{\mathcal{M}}_{g,n}} \Omega^{[u]}_{g,n}(qr,q;qr-a) \prod_{i=1}^{n} \sum_{d_i \geq 0} T_{d_i,a_i} \psi_i^{d_i}. \tag{3}$$

is a **solution of the KP hierarchy** with respect to the variables q_i (identically in u), after the linear triangular change of variables $T_{d,a} = T_{d,a}(u;\vec{q})$, defined for $a = 1,\dots,qr$ recursively by $T_{0,a} = u^{1-\frac{a}{qr}} q_a$ and

$$T_{d,a}(u;\vec{q}) = \sum_{b \geq 1} f_{b,a,q,r}(u,\vec{q}) \frac{\partial}{\partial q_{qr(b-1)+a}} T_{d-1,a}(u;\vec{q}),$$

$$f_{b,a,q,r}(u,\vec{q}) = u^2 \left(b - 1 + \tfrac{a}{qr}\right) q_{qr(b-1)+a} + u\left(2b - 1 + \tfrac{a}{qr}\right) q_{qrb+a} + b q_{qr(b+1)+a}.$$

For instance:

$$T_{1,a} = u^{1-\frac{a}{qr}} \left(u^2 \tfrac{a}{qr} q_a + u(1 + \tfrac{a}{qr}) q_{qr+a} + q_{2qr+a}\right)$$

$$T_{2,a} = u^{1-\frac{a}{qr}} \Big(u^4 (\tfrac{a}{qr})^2 q_a + u^3 \left(1 + 3\tfrac{a}{qr} + 2(\tfrac{a}{qr})^2\right) q_{qr+a} + u^2\left(5 + 6\tfrac{a}{qr} + (\tfrac{a}{qr})^2\right) q_{2qr+a} + u(7 + 3\tfrac{a}{qr}) q_{3qr+a} + 3q_{4qr+a}\Big)$$

Theorem (Giacchetto, L, Norbury)

Let q, r be positive integers. The generating series Chiodo integrals in the formal commuting variables $u, \{T_{d,a}\}_{d \geq 0, a=1,\ldots,qr}$,

$$G^{qr,q}(u;\vec{T}) = \sum_{g,n} \frac{1}{n!} \sum_{a_1,\ldots,a_n=1}^{qr} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}^{[u]}(qr,q;qr-a) \prod_{i=1}^{n} \sum_{d_i \geq 0} T_{d_i,a_i} \psi_i^{d_i}. \tag{3}$$

is a **solution of the KP hierarchy** with respect to the variables q_i (identically in u), after the linear triangular change of variables $T_{d,a} = T_{d,a}(u;\vec{q})$, defined for $a = 1, \ldots, qr$ recursively by $T_{0,a} = u^{1-\frac{a}{qr}} q_a$ and

$$T_{d,a}(u;\vec{q}) = \sum_{b \geq 1} f_{b,a,q,r}(u,\vec{q}) \frac{\partial}{\partial q_{qr(b-1)+a}} T_{d-1,a}(u;\vec{q}),$$

$$f_{b,a,q,r}(u,\vec{q}) = u^2 \left(b - 1 + \tfrac{a}{qr}\right) q_{qr(b-1)+a} + u \left(2b - 1 + \tfrac{a}{qr}\right) q_{qrb+a} + b q_{qr(b+1)+a}.$$

For instance:

$$T_{1,a} = u^{1-\frac{a}{qr}} \left(u^2 \tfrac{a}{qr} q_a + u(1 + \tfrac{a}{qr}) q_{qr+a} + q_{2qr+a}\right)$$

$$T_{2,a} = u^{1-\frac{a}{qr}} \Big(u^4 (\tfrac{a}{qr})^2 q_a + u^3 \left(1 + 3\tfrac{a}{qr} + 2(\tfrac{a}{qr})^2\right) q_{qr+a}$$

$$+ u^2 \left(5 + 6\tfrac{a}{qr} + (\tfrac{a}{qr})^2\right) q_{2qr+a} + u(7 + 3\tfrac{a}{qr}) q_{3qr+a} + 3 q_{4qr+a}\Big)$$

Theorem (Giacchetto, L, Norbury)

Let q, r be positive integers. The generating series Chiodo integrals in the formal commuting variables $u, \{T_{d,a}\}_{d\geq 0, a=1,\ldots,qr}$,

$$G^{qr,q}(u;\vec{T}) = \sum_{g,n} \frac{1}{n!} \sum_{a_1,\ldots,a_n=1}^{qr} \int_{\overline{\mathcal{M}}_{g,n}} \Omega^{[u]}_{g,n}(qr, q; qr-a) \prod_{i=1}^{n} \sum_{d_i\geq 0} T_{d_i,a_i} \psi_i^{d_i}. \tag{3}$$

is a **solution of the KP hierarchy** with respect to the variables q_i (identically in u), after the linear triangular change of variables $T_{d,a} = T_{d,a}(u;\vec{q})$, defined for $a = 1, \ldots, qr$ recursively by $T_{0,a} = u^{1-\frac{a}{qr}} q_a$ and

$$T_{d,a}(u;\vec{q}) = \sum_{b\geq 1} f_{b,a,q,r}(u,\vec{q}) \frac{\partial}{\partial q_{qr(b-1)+a}} T_{d-1,a}(u;\vec{q}),$$

$$f_{b,a,q,r}(u,\vec{q}) = u^2 \left(b - 1 + \tfrac{a}{qr}\right) q_{qr(b-1)+a} + u \left(2b - 1 + \tfrac{a}{qr}\right) q_{qrb+a} + b q_{qr(b+1)+a}.$$

For instance:

$$T_{1,a} = u^{1-\frac{a}{qr}} \left(u^2 \tfrac{a}{qr} q_a + u(1 + \tfrac{a}{qr}) q_{qr+a} + q_{2qr+a}\right)$$

$$\begin{aligned} T_{2,a} = u^{1-\frac{a}{qr}} \Big(& u^4 (\tfrac{a}{qr})^2 q_a + u^3 \left(1 + 3\tfrac{a}{qr} + 2(\tfrac{a}{qr})^2\right) q_{qr+a} \\ & + u^2 \left(5 + 6\tfrac{a}{qr} + (\tfrac{a}{qr})^2\right) q_{2qr+a} + u(7 + 3\tfrac{a}{qr}) q_{3qr+a} + 3q_{4qr+a} \Big) \end{aligned}$$

Corollary

Kazarian Theorem for $r = q = 1$. Witten Conjecture / Kontsevich Theorem for $r = q = 1$ and $u = 0$.

Strategy of the proof:

- The generating function $H^{qr,q}$ of spin-orbifold Hurwitz numbers satisfy KP
- Insert Generalised Zvonkine ELSV
- Set $T_{d,a}(\beta;\vec{p}) = \alpha^{\frac{2d}{3}+\frac{1}{3}} \sum_b \frac{(b+\frac{a}{qr})^{b+d}}{b!} (qr\beta)^{b+\frac{a}{qr}} p_{qrb+a}$.
- Follow the same strategy. Need qr infinite collections of variables that do interact with each other.
- The element representing the β flow is $\mathcal{S}_{qr} = -(2\Lambda_{qr} - J_{qr,qr} + qr\beta\,\Lambda_{2qr} - qr\beta\,J_{qr,2qr}) \in \widehat{\mathfrak{gl}(\infty)}$, where $\Lambda_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\alpha_{-m-k}, \quad J_{\rho,m} = \frac{1}{2}\sum_{k\in\mathbb{Z},\rho\nmid k} \alpha_{\rho,k}\alpha_{\rho,-m-k}$, α being (possibly rescaled) the Bosonic operators acting on the Fock space. In particular, $\mathcal{S}_1 = -(2\Lambda_1 + \beta\Lambda_2)$ Kazarian operator.

Remarks/considerations:

- Integrals of Ω obeys KP.
- Ω can be the CohFT of many enumerative problems in ENT.
- Therefore Ω can transfer its integrability through the spectral curve, *depending* on the spectral curve involved.
- Applications to the integrability of classes constructed via Ω even without being involved in ENT.
- Still unclear for s not dividing r.

Corollary

Kazarian Theorem for $r = q = 1$. Witten Conjecture / Kontsevich Theorem for $r = q = 1$ and $u = 0$.

Strategy of the proof:

- The generating function $H^{qr,q}$ of spin-orbifold Hurwitz numbers satisfy KP
- Insert Generalised Zvonkine ELSV
- Set $T_{d,a}(\beta;\vec{p}) = \alpha^{\frac{2d}{3}+\frac{1}{3}} \sum_b \frac{(b+\frac{a}{qr})^{b+d}}{b!} (qr\beta)^{b+\frac{a}{qr}} p_{qrb+a}$.
- Follow the same strategy. Need qr infinite collections of variables that do interact with each other.
- The element representing the β flow is $\mathcal{S}_{qr} = -(2\Lambda_{qr} - J_{qr,qr} + qr\beta\,\Lambda_{2qr} - qr\beta\,J_{qr,2qr}) \in \widehat{\mathfrak{gl}(\infty)}$, where $\Lambda_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k \alpha_{-m-k}$, $\quad J_{\rho,m} = \frac{1}{2}\sum_{k\in\mathbb{Z},\rho\nmid k} \alpha_{\rho,k}\alpha_{\rho,-m-k}$, α being (possibly rescaled) the Bosonic operators acting on the Fock space. In particular, $\mathcal{S}_1 = -(2\Lambda_1 + \beta\Lambda_2)$ Kazarian operator.

Remarks/considerations:

- Integrals of Ω obeys KP.
- Ω can be the CohFT of many enumerative problems in ENT.
- Therefore Ω can transfer its integrability through the spectral curve, *depending* on the spectral curve involved.
- Applications to the integrability of classes constructed via Ω even without being involved in ENT.
- Still unclear for s not dividing r.

Corollary

Kazarian Theorem for $r = q = 1$. Witten Conjecture / Kontsevich Theorem for $r = q = 1$ and $u = 0$.

Strategy of the proof:

- The generating function $H^{qr,q}$ of spin-orbifold Hurwitz numbers satisfy KP
- Insert Generalised Zvonkine ELSV
- Set $T_{d,a}(\beta;\vec{p}) = \alpha^{\frac{2d}{3}+\frac{1}{3}} \sum_b \frac{(b+\frac{a}{qr})^{b+d}}{b!} (qr\beta)^{b+\frac{a}{qr}} p_{qrb+a}$.
- Follow the same strategy. Need qr infinite collections of variables that do interact with each other.
- The element representing the β flow is $\mathcal{S}_{qr} = -(2\Lambda_{qr} - J_{qr,qr} + qr\beta\,\Lambda_{2qr} - qr\beta\, J_{qr,2qr}) \in \widehat{\mathfrak{gl}(\infty)}$, where $\Lambda_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\alpha_{-m-k}, \quad J_{\rho,m} = \frac{1}{2}\sum_{k\in\mathbb{Z},\rho\nmid k} \alpha_{\rho,k}\alpha_{\rho,-m-k}$, α being (possibly rescaled) the Bosonic operators acting on the Fock space. In particular, $\mathcal{S}_1 = -(2\Lambda_1 + \beta\Lambda_2)$ Kazarian operator.

Remarks/considerations:

- Integrals of Ω obeys KP.
- Ω can be the CohFT of many enumerative problems in ENT.
- Therefore Ω can transfer its integrability through the spectral curve, *depending* on the spectral curve involved.
- Applications to the integrability of classes constructed via Ω even without being involved in ENT.
- Still unclear for s not dividing r.

Corollary

Kazarian Theorem for $r = q = 1$. Witten Conjecture / Kontsevich Theorem for $r = q = 1$ and $u = 0$.

Strategy of the proof:

- The generating function $H^{qr,q}$ of spin-orbifold Hurwitz numbers satisfy KP
- Insert Generalised Zvonkine ELSV
- Set $T_{d,a}(\beta;\vec{p}) = \alpha^{\frac{2d}{3}+\frac{1}{3}} \sum_b \frac{(b+\frac{a}{qr})^{b+d}}{b!}(qr\beta)^{b+\frac{a}{qr}} p_{qrb+a}$.
- Follow the same strategy. Need qr infinite collections of variables that do interact with each other.
- The element representing the β flow is $\mathcal{S}_{qr} = -(2\Lambda_{qr} - J_{qr,qr} + qr\beta\,\Lambda_{2qr} - qr\beta\,J_{qr,2qr}) \in \widehat{\mathfrak{gl}(\infty)}$, where $\Lambda_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\alpha_{-m-k}$, $\quad J_{\rho,m} = \frac{1}{2}\sum_{k\in\mathbb{Z},\rho\nmid k} \alpha_{\rho,k}\alpha_{\rho,-m-k}$, α being (possibly rescaled) the Bosonic operators acting on the Fock space. In particular, $\mathcal{S}_1 = -(2\Lambda_1 + \beta\Lambda_2)$ Kazarian operator.

Remarks/considerations:

- Integrals of Ω obeys KP.
- Ω can be the CohFT of many enumerative problems in ENT.
- Therefore Ω can transfer its integrability through the spectral curve, *depending* on the spectral curve involved.
- Applications to the integrability of classes constructed via Ω even without being involved in ENT.
- Still unclear for s not dividing r.

Corollary

Kazarian Theorem for $r = q = 1$. Witten Conjecture / Kontsevich Theorem for $r = q = 1$ and $u = 0$.

Strategy of the proof:

- The generating function $H^{qr,q}$ of spin-orbifold Hurwitz numbers satisfy KP
- Insert Generalised Zvonkine ELSV
- Set $T_{d,a}(\beta;\vec{p}) = \alpha^{\frac{2d}{3}+\frac{1}{3}} \sum_b \frac{(b+\frac{a}{qr})^{b+d}}{b!} (qr\beta)^{b+\frac{a}{qr}} p_{qrb+a}$.
- Follow the same strategy. Need qr infinite collections of variables that do interact with each other.
- The element representing the β flow is $\mathcal{S}_{qr} = -(2\Lambda_{qr} - J_{qr,qr} + qr\beta\,\Lambda_{2qr} - qr\beta\,J_{qr,2qr}) \in \widehat{\mathfrak{gl}(\infty)}$, where $\Lambda_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\alpha_{-m-k}$, $\quad J_{\rho,m} = \frac{1}{2}\sum_{k\in\mathbb{Z},\rho\nmid k} \alpha_{\rho,k}\alpha_{\rho,-m-k}$, α being (possibly rescaled) the Bosonic operators acting on the Fock space. In particular, $\mathcal{S}_1 = -(2\Lambda_1 + \beta\Lambda_2)$ Kazarian operator.

Remarks/considerations:

- Integrals of Ω obeys KP.
- Ω can be the CohFT of many enumerative problems in ENT.
- Therefore Ω can transfer its integrability through the spectral curve, *depending* on the spectral curve involved.
- Applications to the integrability of classes constructed via Ω even without being involved in ENT.
- Still unclear for s not dividing r.

Corollary

Kazarian Theorem for $r = q = 1$. Witten Conjecture / Kontsevich Theorem for $r = q = 1$ and $u = 0$.

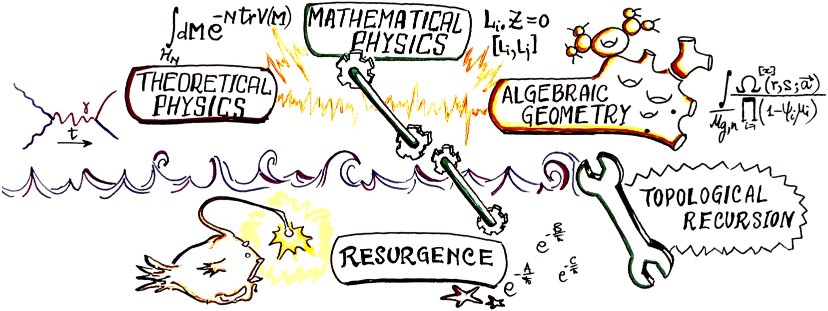
Strategy of the proof:

- The generating function $H^{qr,q}$ of spin-orbifold Hurwitz numbers satisfy KP
- Insert Generalised Zvonkine ELSV
- Set $T_{d,a}(\beta;\vec{p}) = \alpha^{\frac{2d}{3}+\frac{1}{3}} \sum_b \frac{(b+\frac{a}{qr})^{b+d}}{b!} (qr\beta)^{b+\frac{a}{qr}} p_{qrb+a}$.
- Follow the same strategy. Need qr infinite collections of variables that do interact with each other.
- The element representing the β flow is $\mathcal{S}_{qr} = -(2\Lambda_{qr} - J_{qr,qr} + qr\beta\,\Lambda_{2qr} - qr\beta\,J_{qr,2qr}) \in \widehat{\mathfrak{gl}(\infty)}$, where $\Lambda_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\alpha_{-m-k}$, $\quad J_{\rho,m} = \frac{1}{2}\sum_{k\in\mathbb{Z},\rho\nmid k} \alpha_{\rho,k}\alpha_{\rho,-m-k}$, α being (possibly rescaled) the Bosonic operators acting on the Fock space. In particular, $\mathcal{S}_1 = -(2\Lambda_1 + \beta\Lambda_2)$ Kazarian operator.

Remarks/considerations:

- Integrals of Ω obeys KP.
- Ω can be the CohFT of many enumerative problems in ENT.
- Therefore Ω can transfer its integrability through the spectral curve, *depending* on the spectral curve involved.
- Applications to the integrability of classes constructed via Ω even without being involved in ENT.
- Still unclear for s not dividing r.

Chiodo classes Ω lie at the foundations of many enumerative problems: we can now control their integrability explicitly.
We hope this will contribute soon to the understanding of the bigger picture.



What about resurgence?

- We have many exciting work in progress, with Eynard, Garcia-Failde, Gregori, L., Ooms, Schiappa (subsets of), from different perspective and using different methods, for certain classes of enumerative problems.
- Several results already proved
- Hopefully able to report on some of them in a few months, research agenda at least for over a year.

Thank you!

1. J. E. Andersen, G. Borot, S. Charbonnier, V. Delecroix, A. Giacchetto, D. Lewański, and C. Wheeler. *Topological recursion for Masur-Veech volumes.* arXiv:1905.10352.
2. D. Chen, M. Möller, A. Sauvaget, with an appendix by G. Borot, A. Giacchetto, D. Lewański. *Masur-Veech volumes and intersection theory: the principal strata of quadratic differentials.* arXiv:1912.02267.
3. A. Chiodo. *Towards an enumerative geometry of the moduli space of twisted curves and r-th roots.* Compositio Mathematica 144(06).
4. A. Giacchetto, D. Lewański, P. Norbury. *KP Hierarchy for Chiodo integrals.* To appear.
5. M. Kazarian, *KP hierarchy for Hodge integrals.* Advances in Mathematics 221 (2009) 1-21.
6. M. Kazarian, *Recursion for Masur-Veech volumes of moduli spaces of quadratic differentials.* arXiv:1912.10422.
7. D. Lewański, A. Popolitov, S. Shadrin, D. Zvonkine. *Chiodo formulas for the r-th roots and topological recursion* Letters Mathematical Physics, 107(5), 901-919. 2015.
8. D. Yang, D. Zagier, Y. Zhang. *Masur-Veech volumes of quadratic differentials and their asymptotics.* arXiv:2005.02275.