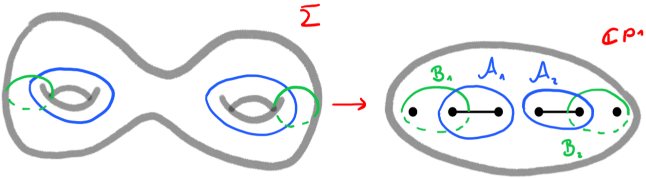
From loop equations to PDEs quantizing hyperelliptic curves

Elba Garcia-Failde

Institut de Physique Théorique (IPhT, Saclay) and Institut des Hautes Études Scientifiques (IHES)

(based on joint work with B. Eynard)



ReNewQuantum Seminar, June 02, 2020

Outline

1. Quantum curves and topological recursion
2. Spectral curves
3. Loop equations and deformation parameters
4. Partition function, wave functions and PDEs
5. Non-perturbative corrections and quantum curve
   - Isomonodromic deformations
6. Questions and future work

Outline

1. Quantum curves and topological recursion
2. Spectral curves
3. Loop equations and deformation parameters
4. Partition function, wave functions and PDEs
5. Non-perturbative corrections and quantum curve
   - Isomonodromic deformations
6. Questions and future work

Presentation of the problem

$P \in \mathbb{C}[x,y]$ and $\Sigma = \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}$ plane curve of genus $\hat{g}$.

A quantization of Σ, is a differential operator $\widehat{P}$ of the form

$$\widehat{P}(\widehat{x},\widehat{y};\hbar) = P_0(\widehat{x},\widehat{y}) + O(\hbar),$$

where $\widehat{x} = x\cdot$, $\widehat{y} = \hbar\frac{\mathrm{d}}{\mathrm{d}x}$, such that $P_0(x,y) = P(x,y)Q(x,y)$, for some $Q \in \mathbb{C}[x,y]$ (often 1).

- The operators $\hat{x}$ and $\hat{y}$ satisfy $[\hat{y},\hat{x}] = \hbar$.
- $\hat{P}(\hat{x},\hat{y})\psi(z,\hbar) = 0$, $z \in \Sigma$.

WKB analysis $\rightsquigarrow \psi(z,\hbar) = \exp\left(\frac{1}{\hbar}S_0(z) + S_1(z) + \hbar S_2(z) + \dots\right)$.

Presentation of the problem

$P \in \mathbb{C}[x,y]$ and $\Sigma = \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}$ plane curve of genus $\hat{g}$.

A quantization of Σ, is a differential operator $\widehat{P}$ of the form

$$\widehat{P}(\widehat{x}, \widehat{y}; \hbar) = P_0(\widehat{x}, \widehat{y}) + O(\hbar),$$

where $\widehat{x} = x \cdot$, $\widehat{y} = \hbar \frac{\mathrm{d}}{\mathrm{d}x}$, such that $P_0(x,y) = P(x,y)Q(x,y)$, for some $Q \in \mathbb{C}[x,y]$ (often 1).

- The operators $\hat{x}$ and $\hat{y}$ satisfy $[\hat{y}, \hat{x}] = \hbar$.
- $\hat{P}(\hat{x}, \hat{y})\psi(z,\hbar) = 0,\ z \in \Sigma$.

WKB analysis $\rightsquigarrow \psi(z,\hbar) = \exp\left(\frac{1}{\hbar} S_0(z) + S_1(z) + \hbar S_2(z) + \dots\right)$.

Question: Can we construct the operator $\hat{P}$ and the solution ψ from P?

Presentation of the problem

$P \in \mathbb{C}[x,y]$ and $\Sigma = \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}$ plane curve of genus $\hat{g}$.

A quantization of Σ, is a differential operator $\widehat{P}$ of the form

$$\widehat{P}(\widehat{x}, \widehat{y}; \hbar) = P_0(\widehat{x}, \widehat{y}) + O(\hbar),$$

where $\widehat{x} = x \cdot$, $\widehat{y} = \hbar \frac{\mathrm{d}}{\mathrm{d}x}$, such that $P_0(x,y) = P(x,y)Q(x,y)$, for some $Q \in \mathbb{C}[x,y]$ (often 1).

- The operators $\hat{x}$ and $\hat{y}$ satisfy $[\hat{y}, \hat{x}] = \hbar$.
- $\hat{P}(\hat{x}, \hat{y})\psi(z, \hbar) = 0$, $z \in \Sigma$.

WKB analysis $\rightsquigarrow \psi(z,\hbar) = \exp\left(\frac{1}{\hbar} S_0(z) + S_1(z) + \hbar S_2(z) + \dots\right)$.

Question: Can we construct the operator $\hat{P}$ and the solution ψ from P?

Conjecture

Both $\hat{P}$ and ψ can be constructed from Σ using topological recursion.

History and literature

- Proved for many particular cases $\rightsquigarrow$ genus $\hat{g} = 0$ spectral curves.
- Bouchard–Eynard '17 $\rightsquigarrow$ spectral curves whose Newton polygon has $N_I := \#\{\text{interior points}\} = 0$ (Fact: $\hat{g} \leq N_I$).

History and literature

- Proved for many particular cases $\rightsquigarrow$ genus $\hat{g} = 0$ spectral curves.
- Bouchard–Eynard '17 $\rightsquigarrow$ spectral curves whose Newton polygon has $N_I := \#\{\text{interior points}\} = 0$ (Fact: $\hat{g} \leq N_I$).
- Mariño–Eynard '08 $\rightsquigarrow$ Holomorphic, modular and background independent, non-perturbative partition functions.
- Borot–Eynard '12 $\rightsquigarrow$ Only non-perturbative wave functions can obey "good" quantum curves (for $\hat{g} > 0$).
- Eynard '17 $\rightsquigarrow$ General idea to construct integrable systems and their τ-functions from the geometry of the spectral curve.

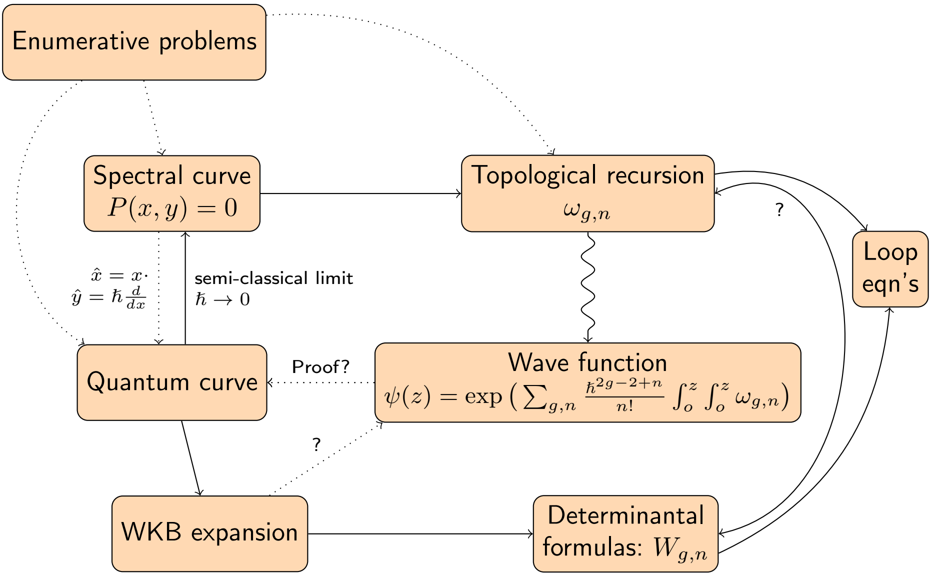
History and literature

- Proved for many particular cases $\rightsquigarrow$ genus $\hat{g} = 0$ spectral curves.
- Bouchard–Eynard '17 $\rightsquigarrow$ spectral curves whose Newton polygon has $N_I := \#\{\text{interior points}\} = 0$ (Fact: $\hat{g} \leq N_I$).
- Mariño–Eynard '08 $\rightsquigarrow$ Holomorphic, modular and background independent, non-perturbative partition functions.
- Borot–Eynard '12 $\rightsquigarrow$ Only non-perturbative wave functions can obey "good" quantum curves (for $\hat{g} > 0$).
- Eynard '17 $\rightsquigarrow$ General idea to construct integrable systems and their τ-functions from the geometry of the spectral curve.
- Chidambaram–Bouchard–Dauphinee '18 $\rightsquigarrow$ Elliptic curves, but with infinitely many $\hbar$ corrections!
- Iwaki–Marchal–Saenz '18, Marchal–Orantin '19 (reversed approach) $\rightsquigarrow$ Lax pairs associated with $\hbar$-dependent Painlevé equations and any $\hbar \partial_x \Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$, with $\mathcal{L}(x,\hbar) \in \mathfrak{sl}_2(\mathbb{C})$, satisfy the topological type property from Bergère–Borot–Eynard '15 ($\hat{g} = 0$).
- Iwaki–Saenz '16, Iwaki '19 $\rightsquigarrow$ Painlevé I and elliptic curves ($\hat{g} = 1$).

History and literature

- Proved for many particular cases $\rightsquigarrow$ genus $\hat{g} = 0$ spectral curves.
- Bouchard–Eynard '17 $\rightsquigarrow$ spectral curves whose Newton polygon has $N_I := \#\{\text{interior points}\} = 0$ (Fact: $\hat{g} \leq N_I$).
- Mariño–Eynard '08 $\rightsquigarrow$ Holomorphic, modular and background independent, non-perturbative partition functions.
- Borot–Eynard '12 $\rightsquigarrow$ Only non-perturbative wave functions can obey "good" quantum curves (for $\hat{g} > 0$).
- Eynard '17 $\rightsquigarrow$ General idea to construct integrable systems and their τ-functions from the geometry of the spectral curve.
- Chidambaram–Bouchard–Dauphinee '18 $\rightsquigarrow$ Elliptic curves, but with infinitely many $\hbar$ corrections!
- Iwaki–Marchal–Saenz '18, Marchal–Orantin '19 (reversed approach) $\rightsquigarrow$ Lax pairs associated with $\hbar$-dependent Painlevé equations and any $\hbar\partial_x\Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$, with $\mathcal{L}(x,\hbar) \in \mathfrak{sl}_2(\mathbb{C})$, satisfy the topological type property from Bergère–Borot–Eynard '15 ($\hat{g} = 0$).
- Iwaki–Saenz '16, Iwaki '19 $\rightsquigarrow$ Painlevé I and elliptic curves ($\hat{g} = 1$).
- Marchal–Orantin '19, Eynard–GF '19 $\rightsquigarrow$ Hyperelliptic (any $\hat{g}$).

Context


Enumerative problems
Spectral curve
$P(x, y) = 0$
Topological recursion
$\omega_{g,n}$
Loop eqn's
?
$\hat{x} = x\cdot$
$\hat{y} = \hbar \frac{d}{dx}$
semi-classical limit
$\hbar \to 0$
Quantum curve
Proof?
Wave function
$\psi(z) = \exp\left(\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_o^z \int_o^z \omega_{g,n} \right)$
?
WKB expansion
Determinantal formulas: $W_{g,n}$

Outline

1. Quantum curves and topological recursion
2. Spectral curves
3. Loop equations and deformation parameters
4. Partition function, wave functions and PDEs
5. Non-perturbative corrections and quantum curve
   - Isomonodromic deformations
6. Questions and future work

Input of topological recursion (TR) (Chekhov–Eynard–Orantin, '04-'07)

Input: *Spectral curve* $\mathcal{S} = (\Sigma, x, y\mathrm{d}x, B)$:

- Σ Riemann surface of genus $\hat{g}$.
- Two meromorphic functions $x, y : \Sigma \to \mathbb{C} \Rightarrow P(x,y) = 0,\ P \in \mathbb{C}[x,y]$.
- Symplectic basis of non-contractible cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{\hat{g}}$ on Σ.
- A symmetric bidifferential $B = \omega_{0,2}$ on $\Sigma \times \Sigma$ such that $\omega_{0,2}(z_1, z_2) \underset{z_2 \to z_1}{\sim} \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}$ + holomorphic with vanishing $\mathcal{A}$-periods.

Output: $\omega_{\mathbf{g},\mathbf{n}}(\mathbf{z_1}, \ldots, \mathbf{z_n}) \in \mathrm{H}^0(\Sigma^n, (K_\Sigma(*\mathrm{Cr}(x)))^{\boxtimes n})^{\mathfrak{S}_n}$, for all $g, n \geq 0$.

Regularity condition: $x : \Sigma \to \mathbb{C}$ meromorphic function with finitely many and simple critical points (denoted $\mathrm{Cr}(x)$), and $y : \Sigma \to \mathbb{C}$ holomorphic on a neighborhood of every $a \in \mathrm{Cr}(x)$ and $\mathrm{d}y(a) \neq 0 \Rightarrow$ Existence of a local involution σ around every ramification point: $x(z) = x(\sigma(z))$.

Generalized cycles

$\mathfrak{M}^1(\Sigma) \rightsquigarrow$ Meromorphic differential forms.

Generalized cycles $\rightsquigarrow \gamma \in \mathfrak{M}^1(\Sigma)^*$ such that $\int_\gamma B$ is a meromorphic 1-form (Eynard '17).

Generalized cycles

$\mathfrak{M}^1(\Sigma) \rightsquigarrow$ Meromorphic differential forms.

Generalized cycles $\rightsquigarrow \gamma \in \mathfrak{M}^1(\Sigma)^*$ such that $\int_\gamma B$ is a meromorphic 1-form (Eynard '17).

- **1st kind cycles:** Usual non-contractible cycles, i.e. elements $[\gamma] \in \mathrm{H}_1(\Sigma, \mathbb{C})$. If Σ is compact of genus g, then $\dim \mathrm{H}_1(\Sigma, \mathbb{C}) = 2g$.
- **2nd kind cycles:** $\gamma = \gamma_p.f \rightsquigarrow$ small circle γ_p around a point $p \in \Sigma$ weighted by a function f holomorphic in a neighborhood of γ_p and meromorphic in a neighborhood of p, with a possible pole at p (of any degree). By definition $\int_\gamma \omega := 2\pi i \operatorname{Res}_p f\omega$.
- **3rd kind cycles:** $\gamma = \gamma_{q\to p} \rightsquigarrow$ paths up to homotopic deformation with fixed endpoints whose boundaries $\partial\gamma = [p] - [q]$ are degree zero divisors.

Parametrization by generalized cycles and times

A basis of functions which are meromorphic in a neighborhood of $p \in \Sigma$ is given by

$$\{\xi_p^k\}_{k\in\mathbb{Z}}, \text{ with } \xi_p = (x - x(p))^{1/\mathrm{ord}_p(x)}.$$

If $x(p) = \infty$, we set $\xi_p = x^{1/\mathrm{ord}_p(x)}$, with $\mathrm{ord}_p(x) < 0$.

Parametrization by generalized cycles and times

A basis of functions which are meromorphic in a neighborhood of $p \in \Sigma$ is given by

$$\{\xi_p^k\}_{k\in\mathbb{Z}}, \text{ with } \xi_p = (x - x(p))^{1/\mathrm{ord}_p(x)}.$$

If $x(p) = \infty$, we set $\xi_p = x^{1/\mathrm{ord}_p(x)}$, with $\mathrm{ord}_p(x) < 0$.
Integer lattice in the space of second kind cycles:

$$\mathcal{A}_{p,k} = \gamma_p.\xi_p^k, \; p \in \Sigma, k \geq 0,$$

$$\mathcal{B}_{p,k} = \frac{1}{2\pi i}\gamma_p.\frac{\xi_p^{-k}}{k}, \; p \in \Sigma, k \geq 1.$$

Parametrization by generalized cycles and times

A basis of functions which are meromorphic in a neighborhood of $p \in \Sigma$ is given by

$$\{\xi_p^k\}_{k\in\mathbb{Z}}, \text{ with } \xi_p = (x - x(p))^{1/\mathrm{ord}_p(x)}.$$

If $x(p) = \infty$, we set $\xi_p = x^{1/\mathrm{ord}_p(x)}$, with $\mathrm{ord}_p(x) < 0$.
Integer lattice in the space of second kind cycles:

$$\mathcal{A}_{p,k} = \gamma_p.\xi_p^k, \; p \in \Sigma, k \geq 0,$$

$$\mathcal{B}_{p,k} = \frac{1}{2\pi i}\gamma_p.\frac{\xi_p^{-k}}{k}, \; p \in \Sigma, k \geq 1.$$

For p a pole of $\omega_{0,1}$:

$$t_{p,k} := \frac{1}{2\pi i}\int_{\mathcal{A}_{p,k}} \omega_{0,1} = \frac{1}{2\pi i}\int_{\gamma_p} \xi_p^k \, \omega_{0,1} = \operatorname*{Res}_p \xi_p^k \omega_{0,1} \,, k \geq 0,$$

$$\frac{\partial}{\partial t_{p,k}}\omega_{0,0} = \frac{1}{2\pi i}\int_{\mathcal{B}_{p,k}} \omega_{0,1} = \frac{1}{2\pi i}\int_{\gamma_p} \frac{\xi_p^{-k}}{k} \, \omega_{0,1} = \operatorname*{Res}_p \frac{\xi_p^{-k}}{k}\omega_{0,1} \,, k \geq 1.$$

Parametrization by generalized cycles and times

A basis of functions which are meromorphic in a neighborhood of $p \in \Sigma$ is given by

$$\{\xi_p^k\}_{k\in\mathbb{Z}}, \text{ with } \xi_p = (x - x(p))^{1/\mathrm{ord}_p(x)}.$$

If $x(p) = \infty$, we set $\xi_p = x^{1/\mathrm{ord}_p(x)}$, with $\mathrm{ord}_p(x) < 0$.
Integer lattice in the space of second kind cycles:

$$\mathcal{A}_{p,k} = \gamma_p.\xi_p^k,\ p \in \Sigma, k \geq 0,$$

$$\mathcal{B}_{p,k} = \frac{1}{2\pi i}\gamma_p.\frac{\xi_p^{-k}}{k},\ p \in \Sigma, k \geq 1.$$

For p a pole of $\omega_{0,1}$:

$$t_{p,k} := \frac{1}{2\pi i}\int_{\mathcal{A}_{p,k}} \omega_{0,1} = \frac{1}{2\pi i}\int_{\gamma_p} \xi_p^k\, \omega_{0,1} = \operatorname*{Res}_p \xi_p^k \omega_{0,1}\, , k \geq 0,$$

$$\frac{\partial}{\partial t_{p,k}}\omega_{0,0} = \frac{1}{2\pi i}\int_{\mathcal{B}_{p,k}} \omega_{0,1} = \frac{1}{2\pi i}\int_{\gamma_p} \frac{\xi_p^{-k}}{k}\, \omega_{0,1} = \operatorname*{Res}_p \frac{\xi_p^{-k}}{k}\omega_{0,1}\, , k \geq 1.$$

Filling fractions: $\varepsilon_i := \frac{1}{2\pi i}\int_{\mathcal{A}_i} \omega_{0,1}.$

Hyperelliptic curves

$$P(x,y) = R(x) - y^2 = 0, \text{ with } R(x) \in \mathbb{C}(x)$$

$x : \Sigma \to \mathbb{CP}^1$ is a double cover and we have a global involution

$$(x,y) \mapsto (x,-y).$$

Hyperelliptic curves

$$P(x,y) = R(x) - y^2 = 0, \text{ with } R(x) \in \mathbb{C}(x)$$

$x : \Sigma \to \mathbb{CP}^1$ is a double cover and we have a global involution

$$(x,y) \mapsto (x,-y).$$

Remark

- *If $R \in \mathbb{C}[x]$ is a polynomial of degree $2m+1$ or $2m+2$ the curve has genus $\hat{g} \le m$, with equality if the plane curve is smooth. If the degree is odd, the curve has one point at infinity and if the degree is even, the curve has two points at infinity.*
- *All curves of genus 2 are hyperelliptic, but for genus ≥ 3 the generic curve is not hyperelliptic.*

$\zeta_i \rightsquigarrow$ pole of $\omega_{0,1}$, $d_i := \mathrm{ord}_{\zeta_i}(x)$.

- $x(\zeta_i) \neq \infty \rightsquigarrow$ Not a ramification point $\Rightarrow d_i = 1$.
- $x(\zeta_i) = \infty \rightsquigarrow$ Either ζ_i is a pole of $\omega_{0,1}$ of odd degree, and then it is a ramification point with $d_i = -2$ and $\sigma(\zeta_i) = \zeta_i$, or it is a pole of $\omega_{0,1}$ of even degree, and then it is not a ramification point with $d_i = -1$.

If ζ_i is a pole, $\sigma(\zeta_i)$ is also a pole. For poles for which $\sigma(\zeta_i) \neq \zeta_i$, we have $t_{\sigma(\zeta_i),j} = -t_{\zeta_i,j}$.

$\zeta_i \rightsquigarrow$ pole of $\omega_{0,1}$, $d_i := \mathrm{ord}_{\zeta_i}(x)$.

- $x(\zeta_i) \neq \infty \rightsquigarrow$ Not a ramification point $\Rightarrow d_i = 1$.
- $x(\zeta_i) = \infty \rightsquigarrow$ Either ζ_i is a pole of $\omega_{0,1}$ of odd degree, and then it is a ramification point with $d_i = -2$ and $\sigma(\zeta_i) = \zeta_i$, or it is a pole of $\omega_{0,1}$ of even degree, and then it is not a ramification point with $d_i = -1$.

If ζ_i is a pole, $\sigma(\zeta_i)$ is also a pole. For poles for which $\sigma(\zeta_i) \neq \zeta_i$, we have $t_{\sigma(\zeta_i),j} = -t_{\zeta_i,j}$.

Parametrizing $\omega_{0,1}$ with $\mathcal{A}_{p,k}$ and $\mathcal{B}_{p,k}$ cycles:

Expansion of $\omega_{0,1}(z) = y(z)dx(z)$, $z \in \Sigma$, around a pole p:

$$\omega_{0,1}(z) = \sum_{k=0}^{s_p} \left(\operatorname*{Res}_{z=p} \zeta_p^k \omega_{0,1}(z) \right) \zeta_p^{-k-1} d\zeta_p + \sum_{k\geq 1} \left(\operatorname*{Res}_{z=p} \zeta_p^{-k} \omega_{0,1}(z) \right) \zeta_p^{k-1} d\zeta_p$$

$$= \sum_{k=0}^{s_p} t_{p,k} \zeta_p^{-k-1} d\zeta_p + \sum_{k\geq 1} k \frac{\partial}{\partial t_{p,k}} \omega_{0,0}\, \zeta_p^{k-1} d\zeta_p.$$

Spectral curves from integrable systems

Definition

Let $\hbar\frac{\partial}{\partial x}\Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$ be a differential system. We define the classical spectral curve associated to it by

$$P(x,y) := \lim_{\hbar\to 0}\det(y\mathrm{Id} - \mathcal{L}(x,\hbar)) = 0,$$

which gives a polynomial equation. For a non-zero genus curve, this must be completed with a choice of symplectic basis of cycles and a bidifferential B.

Spectral curves from integrable systems

Definition

Let $\hbar \frac{\partial}{\partial x} \Psi(x,\hbar) = \mathcal{L}(x,\hbar)\Psi(x,\hbar)$ be a differential system. We define the classical spectral curve associated to it by

$$P(x,y) := \lim_{\hbar \to 0} \det(y\mathrm{Id} - \mathcal{L}(x,\hbar)) = 0,$$

which gives a polynomial equation. For a non-zero genus curve, this must be completed with a choice of symplectic basis of cycles and a bidifferential B.

Different approach:

- $\hbar$-differential system.
- Define the classical spectral curve associated to it.
- Show that interesting quantities from the point of view of the differential system may be reconstructed from topological recursion applied to this classical spectral curve.
- Proof by showing that the differential system satisfies the topological type property (Bergère–Borot–Eynard '15).

Isomonodromic deformations

$\hbar$-dependent Lax pair $(\mathcal{L}(x,t;\hbar),\mathcal{R}(x,t;\hbar)) \rightsquigarrow 2\times 2$ matrices, whose entries are rational functions of x and holomorphic in t such that

$$\begin{cases} \hbar\dfrac{\partial}{\partial x}\Psi(x,t) = \mathcal{L}(x,t;\hbar)\Psi(x,t), \\ \hbar\dfrac{\partial}{\partial t}\Psi(x,t) = \mathcal{R}(x,t;\hbar)\Psi(x,t) \end{cases}$$

is compatible. We call such a system an isomonodromy system.

$\frac{\partial^2}{\partial t\partial x}\Psi = \frac{\partial^2}{\partial x\partial t}\Psi \Leftrightarrow \hbar\frac{\partial\mathcal{L}}{\partial t} - \hbar\frac{\partial\mathcal{R}}{\partial x} + [\mathcal{L},\mathcal{R}] = 0$ (zero-curvature equation).

Isomonodromic deformations

$\hbar$-dependent Lax pair $(\mathcal{L}(x,t;\hbar),\mathcal{R}(x,t;\hbar)) \rightsquigarrow 2\times 2$ matrices, whose entries are rational functions of x and holomorphic in t such that

$$\begin{cases} \hbar\dfrac{\partial}{\partial x}\Psi(x,t) = \mathcal{L}(x,t;\hbar)\Psi(x,t), \\ \hbar\dfrac{\partial}{\partial t}\Psi(x,t) = \mathcal{R}(x,t;\hbar)\Psi(x,t) \end{cases}$$

is compatible. We call such a system an isomonodromy system.

$\frac{\partial^2}{\partial t\partial x}\Psi = \frac{\partial^2}{\partial x\partial t}\Psi \Leftrightarrow \hbar\frac{\partial \mathcal{L}}{\partial t} - \hbar\frac{\partial \mathcal{R}}{\partial x} + [\mathcal{L},\mathcal{R}] = 0$ (zero-curvature equation).

$\mathcal{L}(x,t;\hbar) = \sum_{k\geq 0}\hbar^k\mathcal{L}_k(x,t)$. The associated *spectral curve* is given by

$\det(y\,\mathrm{Id} - \mathcal{L}_0(x,t)) = 0$, family of algebraic curves parametrized by t.

Isomonodromic deformations

$\hbar$-dependent Lax pair $(\mathcal{L}(x,t;\hbar),\mathcal{R}(x,t;\hbar)) \rightsquigarrow 2\times 2$ matrices, whose entries are rational functions of x and holomorphic in t such that

$$\begin{cases} \hbar\dfrac{\partial}{\partial x}\Psi(x,t) = \mathcal{L}(x,t;\hbar)\Psi(x,t), \\ \hbar\dfrac{\partial}{\partial t}\Psi(x,t) = \mathcal{R}(x,t;\hbar)\Psi(x,t) \end{cases}$$

is compatible. We call such a system an isomonodromy system.

$\frac{\partial^2}{\partial t\partial x}\Psi = \frac{\partial^2}{\partial x\partial t}\Psi \Leftrightarrow \hbar\frac{\partial\mathcal{L}}{\partial t} - \hbar\frac{\partial\mathcal{R}}{\partial x} + [\mathcal{L},\mathcal{R}] = 0$ (zero-curvature equation).

$\mathcal{L}(x,t;\hbar) = \sum_{k\geq 0}\hbar^k\mathcal{L}_k(x,t)$. The associated *spectral curve* is given by

$$\det(y\,\mathrm{Id} - \mathcal{L}_0(x,t)) = 0, \text{ family of algebraic curves parametrized by } t.$$

Remark

Painlevé equations $\rightsquigarrow$ *Isomonodromic deformations. Painlevé property* $\rightsquigarrow$ *Solutions have no movable singularities other than poles. Classification of all second order differential equations with the Painlevé property* $\rightsquigarrow 50$ *solutions and only* 6 *which could not be integrated from already known functions.*

Outline

1. Quantum curves and topological recursion
2. Spectral curves
3. Loop equations and deformation parameters
4. Partition function, wave functions and PDEs
5. Non-perturbative corrections and quantum curve
   - Isomonodromic deformations
6. Questions and future work

Loop equations

Let $y^2 = R(x)$, with $R \in \mathbb{C}(x)$. The family of curves that we consider has the global involution $z \mapsto -z$, i.e. $x(z) = x(-z)$.
Let $\omega_{0,1}(z) := y(z)\mathrm{d}x(z)$, $\omega_{0,2}(z_1, z_2) := B(z_1, z_2)$ and $\omega_{g,n}$ for $2g - 2 + n > 0$ be defined as the topological recursion amplitudes for this initial data.

Theorem (Eynard–Orantin, '07)

- *The linear loop equation reads*

$$\omega_{g,n+1}(z, z_1, \ldots, z_n) + \omega_{g,n+1}(-z, z_1, \ldots, z_n) = \delta_{g,0}\delta_{n,1} \frac{\mathrm{d}x(z)\mathrm{d}x(z_1)}{(x(z)-x(z_1))^2}.$$

- *The quadratic loop equations claim that the following expression*

$$\frac{1}{\mathrm{d}x(z)^2}\Bigg(\omega_{g-1,n+2}(z, -z, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g,\\ I_1 \sqcup I_2 = \{z_1,\ldots,z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(-z, I_2)\Bigg)$$

is a rational function of $x(z)$ with no poles at the branch-points.

Corollary

For all $g, n \geq 0$,

$$P_{g,n}(x(z); z_1, \ldots, z_n) := \frac{-1}{\mathrm{d}x(z)^2}\Bigg(\omega_{g-1,n+2}(z, -z, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g, \\ I_1 \sqcup I_2 = \{z_1,\ldots,z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(-z, I_2)\Bigg) + \sum_{i=1}^{n} \mathrm{d}_i \left(\frac{1}{x(z) - x(z_i)} \frac{\omega_{g,n}(z_1, \ldots, -z_i, \ldots, z_n)}{\mathrm{d}x(z_i)}\right)$$

is a rational function of $x(z)$ *that has no poles at the branch-points and no poles when* $x(z) = x(z_i)$.

Corollary

For all $g, n \geq 0$,

$$P_{g,n}(x(z); z_1, \ldots, z_n) := \frac{-1}{\mathrm{d}x(z)^2}\Bigg(\omega_{g-1,n+2}(z, -z, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g,\\ I_1 \sqcup I_2 = \{z_1,\ldots,z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\,\omega_{g_2,1+|I_2|}(-z, I_2)\Bigg) + \sum_{i=1}^{n} \mathrm{d}_i \left(\frac{1}{x(z)-x(z_i)} \frac{\omega_{g,n}(z_1,\ldots,-z_i,\ldots,z_n)}{\mathrm{d}x(z_i)}\right)$$

is a rational function of $x(z)$ *that has no poles at the branch-points and no poles when* $x(z) = x(z_i)$.

- $\omega_{0,2}$ poles at coinciding points.
- $\omega_{g,n}$, $2g-2+n>0$, poles at ramification points.

$\Rightarrow$ $P_{g,n}$, as a function of z, can only have poles at the poles of $\omega_{0,1}$.

Elliptic curves

Consider a family of elliptic curves $y^2 = x^3 + tx + V$ parametrized by

$$\begin{cases} x(z) = \nu^2 \wp(z), \\ y(z) = \frac{\nu^3}{2} \wp'(z), \end{cases}$$

with $z \in \Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ the torus of modulus τ.
Near $z = 0$, $x(z) = \infty$, we have

$$y \sim x^{\frac{3}{2}} + \frac{t}{2x^{\frac{1}{2}}} + \frac{V}{2x^{\frac{3}{2}}} - \frac{t^2}{8x^{\frac{5}{2}}} - \frac{tV}{4x^{\frac{7}{2}}} + O(x^{-\frac{9}{2}})$$

Elliptic curves

Consider a family of elliptic curves $y^2 = x^3 + tx + V$ parametrized by

$$\begin{cases} x(z) = \nu^2 \wp(z), \\ y(z) = \frac{\nu^3}{2} \wp'(z), \end{cases}$$

with $z \in \Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ the torus of modulus τ.
Near $z = 0$, $x(z) = \infty$, we have

$$y \sim x^{\frac{3}{2}} + \frac{t}{2x^{\frac{1}{2}}} + \frac{V}{2x^{\frac{3}{2}}} - \frac{t^2}{8x^{\frac{5}{2}}} - \frac{tV}{4x^{\frac{7}{2}}} + O(x^{-\frac{9}{2}})$$

$\Rightarrow$

$$t_{\infty,1} = \frac{1}{2\pi i} \int_{\mathcal{A}_{\infty,1}} y dx = \operatorname*{Res}_{z\to 0} x^{-\frac{1}{2}} \, y dx = -t,$$

$$\int_{\mathcal{B}_{\infty,1}} y dx = \operatorname*{Res}_{z\to 0} x^{\frac{1}{2}} \, y dx = -V.$$

$$t_{\infty,5} = \frac{1}{2\pi i} \int_{\mathcal{A}_{\infty,5}} y dx = \operatorname*{Res}_{z\to \infty} x^{-\frac{5}{2}} \, y dx = -2,$$

$$\int_{\mathcal{B}_{\infty,5}} y dx = \frac{1}{5} \operatorname*{Res}_{z\to \infty} x^{\frac{5}{2}} \, y dx = \frac{tV}{10}.$$

Genus 1 case

$P_{0,0}(x(z)) = y(z)^2 = x^3 + tx + V$, with

$$-V = \int_{\mathcal{B}_{\infty,1}} \omega_{0,1} = \frac{\partial}{\partial t_{\infty,1}} \omega_{0,0} = -\frac{\partial}{\partial t} \omega_{0,0}$$

$\Rightarrow P_{0,0}(x(z)) = x^3 + tx + \frac{\partial}{\partial t} \omega_{0,0}$.

Genus 1 case

$P_{0,0}(x(z)) = y(z)^2 = x^3 + tx + V$, with

$$-V = \int_{\mathcal{B}_{\infty,1}} \omega_{0,1} = \frac{\partial}{\partial t_{\infty,1}} \omega_{0,0} = -\frac{\partial}{\partial t} \omega_{0,0}$$

$\Rightarrow P_{0,0}(x(z)) = x^3 + tx + \frac{\partial}{\partial t} \omega_{0,0}$.

For $2g - 2 + n \geq 0$, for $z \to 0$ ($x(z) \to \infty$) we have

$$\begin{aligned} P_{g,n}(x(z); z_1, \ldots, z_n) &= 2 \frac{y(z) \mathrm{d}x(z)\, \omega_{g,n+1}(z, z_1, \ldots, z_n)}{\mathrm{d}x(z)^2} + O(x(z)^{-3}), \\ &= 2y(z) O(x(z)^{\frac{-3}{2}}) + O(x(z)^{-3}) = O(1). \end{aligned}$$

Genus 1 case

$P_{0,0}(x(z)) = y(z)^2 = x^3 + tx + V$, with

$$-V = \int_{\mathcal{B}_{\infty,1}} \omega_{0,1} = \frac{\partial}{\partial t_{\infty,1}} \omega_{0,0} = -\frac{\partial}{\partial t} \omega_{0,0}$$

$\Rightarrow P_{0,0}(x(z)) = x^3 + tx + \frac{\partial}{\partial t}\omega_{0,0}$.

For $2g-2+n \geq 0$, for $z \to 0$ ($x(z) \to \infty$) we have

$$\begin{aligned} P_{g,n}(x(z); z_1, \dots, z_n) &= 2\frac{y(z)\mathrm{d}x(z)\,\omega_{g,n+1}(z, z_1, \dots, z_n)}{\mathrm{d}x(z)^2} + O(x(z)^{-3}), \\ &= 2y(z) O(x(z)^{\frac{-3}{2}}) + O(x(z)^{-3}) = O(1). \end{aligned}$$

So $P_{g,n}$ is independent of z and we have:

$$\begin{aligned} P_{g,n}(x(z); z_1, \dots, z_n) &= 2 \lim_{x(z)\to\infty} x(z)^{\frac{3}{2}} \frac{\omega_{g,n+1}(z, z_1, \dots, z_n)}{\mathrm{d}x(z)} \\ &= -\operatorname*{Res}_{z\to\infty} \sqrt{x(z)}\,\omega_{g,n+1}(z, z_1, \dots, z_n) \\ &= -\oint_{\mathcal{B}_{\infty,1}} \omega_{g,n+1}(z, z_1, \dots, z_n) = \frac{\partial}{\partial t}\omega_{g,n}(z_1, \dots, z_n). \end{aligned}$$

Generalized cycles and variational formulas

Proposition

Let ζ_i *be the poles of* $\omega_{0,1}$ *of degrees* $m_i + 1$ *and* $d_i := \mathrm{ord}_{\zeta_i}(x)$.
Consider the operator

$$L(x) := \sum_{i, x(\zeta_i)=\infty} \sum_{j=1-2d_i}^{m_i} t_{\zeta_i,j} \sum_{0 \leq k \leq \frac{1-j}{d_i}-2} x^k \left(-\frac{j}{d_i} - k - 2 \right) \frac{\partial}{\partial t_{\zeta_i, j+d_i(k+2)}}$$

$$+ \sum_{i, x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} (x - x(\zeta_i))^{-(k+1)} (j+1-k) \frac{\partial}{\partial t_{\zeta_i, j+1-k}},$$

$$\Rightarrow P_{g,n}(x; z_1, \ldots, z_n) = L(x).\omega_{g,n}(z_1, \ldots, z_n).$$

Example

In the Airy case, $y^2 = x$, we have only one pole, at $\zeta_i = \infty$, of degree $m_i = 3$, with $d_i = -2$. The sum is empty and $L(x) = 0$.

Outline

1. Quantum curves and topological recursion
2. Spectral curves
3. Loop equations and deformation parameters
4. Partition function, wave functions and PDEs
5. Non-perturbative corrections and quantum curve
   - Isomonodromic deformations
6. Questions and future work

Wave function over a divisor

Let $D = \sum_{i=1}^{r} \alpha_i [p_i]$ divisor on Σ, with $p_i \in \Sigma$. Call $\sum_i \alpha_i$ the degree of the divisor. For $D \in \mathrm{Div}_0(\Sigma)$ (divisors of degree 0), $\rho(z)$ 1-form on Σ and $o \in \Sigma$ an arbitrary base point:

$$\int_D \rho(z) := \sum_i \alpha_i \int_o^{p_i} \rho(z).$$

Wave function over a divisor

Let $D = \sum_{i=1}^{r} \alpha_i [p_i]$ divisor on Σ, with $p_i \in \Sigma$. Call $\sum_i \alpha_i$ the degree of the divisor. For $D \in \mathrm{Div}_0(\Sigma)$ (divisors of degree 0), $\rho(z)$ 1-form on Σ and $o \in \Sigma$ an arbitrary base point:

$$\int_D \rho(z) := \sum_i \alpha_i \int_o^{p_i} \rho(z).$$

For $(g,n) \neq (0,2)$ consider the functions of D, defined locally:

$$F_{g,0}(D) := F_g = \omega_{g,0},$$

$$F_{g,n}(D) := \overbrace{\int_D \cdots \int_D}^{n} \omega_{g,n}(z_1, \dots, z_n).$$

$$B(z_1, z_2) := \mathrm{d}_1 \mathrm{d}_2 \log \left(E(z_1, z_2) \sqrt{\mathrm{d}x(z_1) \mathrm{d}x(z_2)} \right),$$

with $E(z_1, z_2)$ being the prime form, which vanishes only if $z_1 = z_2$ with a simple zero and has no pole.

Special regularization for cylinders

For $(g,n) = (0,2)$ define:

$$F_{0,2}(D) := 2\sum_{i<j} \alpha_i \alpha_j \log\left(E(p_i,p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)} \right).$$

Special regularization for cylinders

For $(g,n)=(0,2)$ define:

$$F_{0,2}(D) := 2\sum_{i<j}\alpha_i\alpha_j \log\left(E(p_i,p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)}\right).$$

More usual regularization for $(0,2)$:

$$\tilde{F}_{0,2}(D) := \int_D\int_D \left(B(z_1,z_2) - \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right).$$

Special regularization for cylinders

For $(g,n)=(0,2)$ define:

$$F_{0,2}(D) := 2\sum_{i<j}\alpha_i\alpha_j \log\left(E(p_i,p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)}\right).$$

More usual regularization for $(0,2)$:

$$\tilde{F}_{0,2}(D) := \int_D\int_D \left(B(z_1,z_2) - \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right).$$

We have

$$\tilde{F}_{0,2}(D) = 2\sum_{i<j}\alpha_i\alpha_j\log\left(\frac{E(p_i,p_j)\sqrt{\mathrm{d}x_i\mathrm{d}x_j}}{x_i-x_j}\right) + \sum_i \alpha_i^2 \log\frac{\mathrm{d}x_i}{\mathrm{d}x_i}.$$

Then $F_{0,2}(D) = \tilde{F}_{0,2}(D) + 2\sum_{i<j}\alpha_i\alpha_j\log(x_i-x_j)$.

Perturbative PDE as a quantization

Define

$$S_m(D) := \sum_{\substack{2g-2+n=m-1 \\ g\geq 0, n\geq 1}} \frac{F_{g,n}(D)}{n!},$$

and the wave function

$$\psi(D,\hbar) := \exp(S(D,\hbar)), \text{ with } S(D,\hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(D).$$

Perturbative PDE as a quantization

Define

$$S_m(D) := \sum_{\substack{2g-2+n=m-1 \\ g\geq 0, n\geq 1}} \frac{F_{g,n}(D)}{n!},$$

and the wave function

$$\psi(D,\hbar) := \exp(S(D,\hbar)), \text{ with } S(D,\hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(D).$$

Theorem (Eynard–GF,'19)

Let $F := \sum_{g>0} \hbar^{2g} F_g$*. For every* $k = 1, \dots, r$*, we obtain*

$$\hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_k^2} - \sum_{i\neq k} \frac{\frac{\mathrm{d}}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_k}\frac{\mathrm{d}}{\mathrm{d}x_k}}{x_k - x_i} - L(x_k) + \sum_{\substack{i\neq j \\ i\neq k, j\neq k}} \frac{\alpha_i\alpha_j}{(x_k - x_i)(x_i - x_j)} \right)\psi$$

$$= (R(x_k) + L(x_k).F)\,\psi.$$

Perturbative PDE as a quantization

Define

$$S_m(D) := \sum_{\substack{2g-2+n=m-1 \\ g\geq 0, n\geq 1}} \frac{F_{g,n}(D)}{n!},$$

and the wave function

$$\psi(D,\hbar) := \exp(S(D,\hbar)), \text{ with } S(D,\hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(D).$$

Theorem (Eynard–GF,'19)

Let $F := \sum_{g>0} \hbar^{2g} F_g$. For every $k = 1,\dots,r$, we obtain

$$\hbar^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x_k^2} - \sum_{i\neq k} \frac{\frac{\mathrm{d}}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_k}\frac{\mathrm{d}}{\mathrm{d}x_k}}{x_k - x_i} - L(x_k) + \sum_{\substack{i\neq j \\ i\neq k, j\neq k}} \frac{\alpha_i\alpha_j}{(x_k-x_i)(x_i-x_j)}\right)\psi$$

$$= (R(x_k) + L(x_k).F)\,\psi.$$

Idea of the proof: $\int_D \cdots \int_D P_{g,n}(x(z); z_1,\dots,z_n) = L(x).F_{g,n}(D)$ and take the limit $z \to p_k$.

Airy and elliptic cases for two-point divisors

Divisor $D = [z_1] - [z_2]$:

- PDE for Airy curve: $y^2 = x$. We had $P_{g,n} = 0$.

$$\begin{cases} \hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi & = x_1 \psi, \\ \hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_2^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi & = x_2 \psi. \end{cases}$$

Airy and elliptic cases for two-point divisors

Divisor $D = [z_1] - [z_2]$:

- PDE for Airy curve: $y^2 = x$. We had $P_{g,n} = 0$.

$$\begin{cases} \hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi & = x_1 \psi, \\ \hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_2^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi & = x_2 \psi. \end{cases}$$

- PDE for elliptic curve: We had $P_{g,n} = \frac{\partial}{\partial t} \omega_{g,n}(z_1, \ldots, z_n)$.

$$\left(\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x_k^2} + \hbar^2 \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} - \frac{\partial}{\partial t} \right) \psi(D) = \left(x_k^3 + t x_k + V + \frac{\partial}{\partial t} F \right) \psi,$$

for $k = 1, 2$.

Airy and elliptic cases for two-point divisors

Divisor $D = [z_1] - [z_2]$:

- PDE for Airy curve: $y^2 = x$. We had $P_{g,n} = 0$.

$$\begin{cases} \hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi & = x_1 \psi, \\ \hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_2^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} \right) \psi & = x_2 \psi. \end{cases}$$

- PDE for elliptic curve: We had $P_{g,n} = \frac{\partial}{\partial t} \omega_{g,n}(z_1, \ldots, z_n)$.

$$\left(\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x_k^2} + \hbar^2 \frac{\frac{\mathrm{d}}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_2}}{x_1 - x_2} - \frac{\partial}{\partial t} \right) \psi(D) = \left(x_k^3 + t x_k + V + \frac{\partial}{\partial t} F \right) \psi,$$

for $k = 1, 2$.

Remark

For a 2-point divisor, we have a PDE involving $\mathrm{d}/\mathrm{d}x_1$, $\mathrm{d}/\mathrm{d}x_2$ and partial derivatives with respect to times when $L(x) \neq 0$. It is possible to eliminate $\mathrm{d}/\mathrm{d}x_2$ and get a PDE involving only $\mathrm{d}/\mathrm{d}x_1$ and time derivatives. What about eliminating the time derivatives?

Outline

1. Quantum curves and topological recursion
2. Spectral curves
3. Loop equations and deformation parameters
4. Partition function, wave functions and PDEs
5. Non-perturbative corrections and quantum curve
   - Isomonodromic deformations
6. Questions and future work

Non-perturbative wave-functions and quantum curve

Problem for genus $\hat{g} > 0$: $\int_o^z \cdots \int_o^z \omega_{g,n}$ are not invariant after z goes around a cycle. (Borot–Eynard,'12)

Non-perturbative wave-functions and quantum curve

Problem for genus $\hat{g} > 0$: $\int_o^z \cdots \int_o^z \omega_{g,n}$ are not invariant after z goes around a cycle. (Borot–Eynard,'12)
Since our PDEs do not involve the (first kind) times ε_i, shifting the cycles $\mathcal{A}_i$ by integer cycles in $\mathrm{H}_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2\hat{g}}$ gives another solution:

$$\psi(\{\mathcal{A}_i \to \mathcal{A}_i + n_i; [z] - [z']).$$

Non-perturbative wave-functions and quantum curve

Problem for genus $\hat{g} > 0$: $\int_o^z \cdots \int_o^z \omega_{g,n}$ are not invariant after z goes around a cycle. (Borot–Eynard,'12)
Since our PDEs do not involve the (first kind) times ε_i, shifting the cycles $\mathcal{A}_i$ by integer cycles in $\mathrm{H}_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2\hat{g}}$ gives another solution:

$$\psi(\{\mathcal{A}_i \to \mathcal{A}_i + n_i; [z] - [z']).$$

The transseries linear combination

$$\hat{\psi}([z]-[z']) = \frac{1}{\mathcal{T}} \sum_{n_1,\dots,n_{\hat{g}} \in \mathbb{Z}^{\hat{g}}} \psi(\{\varepsilon_i \to \varepsilon_i + n_i\}; [z]-[z'])Z(\{\varepsilon_i \to \varepsilon_i + n_i\}),$$

where

$$\mathcal{T} = \sum_{n_1,\dots,n_{\hat{g}} \in \mathbb{Z}^{\hat{g}}} Z(\{\varepsilon_i \to \varepsilon_i + n_i\}),$$

is thus also a solution of the same PDE.

Outline

1. Quantum curves and topological recursion
2. Spectral curves
3. Loop equations and deformation parameters
4. Partition function, wave functions and PDEs
5. Non-perturbative corrections and quantum curve
   - Isomonodromic deformations
6. Questions and future work

Isomonodromic systems and ODEs

Goal: $\hat{\psi}$ obeys an isomonodromic system type of equation.
Method: Prove that $\hat{\psi}([z]-[z'])$ coincides with the integrable kernel of an isomonodromic system.

Isomonodromic systems and ODEs

Goal: $\hat{\psi}$ obeys an isomonodromic system type of equation.
Method: Prove that $\hat{\psi}([z]-[z'])$ coincides with the integrable kernel of an isomonodromic system.

Assume that there is an isomonodromic system $(\mathcal{L}, \mathcal{R})$ whose associated spectral curve is our spectral curve, with

$$\Psi(x) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}$$

a formal transseries solution. The integrable kernel is defined as:

$$K(x,x') \coloneqq \frac{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}{x-x'}.$$

Theorem (Eynard–GF '19)

$$\psi([z]-[z']) = \frac{A(x(z))\tilde{B}(x(z')) - \tilde{A}(x(z))B(x(z'))}{x(z)-x(z')}.$$

ODE from isomonodromic system

Idea of the proof:

- Show that the ratio of $\hat{\psi}$ and K has to be a formal series of the form $1 + O(z'^{-1})$.
- Imposing the reduced PDE, deduce a PDE for the ratio.
- Being a solution of this PDE, the ratio must be 1.

ODE from isomonodromic system

Idea of the proof:

- Show that the ratio of $\hat{\psi}$ and K has to be a formal series of the form $1 + O(z'^{-1})$.
- Imposing the reduced PDE, deduce a PDE for the ratio.
- Being a solution of this PDE, the ratio must be 1.

Let ζ be such that $x(\zeta) = \infty$. The theorem implies that

$$\lim_{z' \to \zeta} \frac{(x(z) - x(z'))\psi([z] - [z'])}{\tilde{B}(x(z'))} = A(x(z)).$$

ODE from isomonodromic system

Idea of the proof:

- Show that the ratio of $\hat{\psi}$ and K has to be a formal series of the form $1+O(z'^{-1})$.
- Imposing the reduced PDE, deduce a PDE for the ratio.
- Being a solution of this PDE, the ratio must be 1.

Let ζ be such that $x(\zeta)=\infty$. The theorem implies that

$$\lim_{z'\to\zeta}\frac{(x(z)-x(z'))\psi([z]-[z'])}{\tilde{B}(x(z'))}=A(x(z)).$$

Since $A(x), \tilde{A}(x)$ satisfy the isomonodromic system

$$\hbar\frac{\partial}{\partial x}\begin{pmatrix}A(x)\\ \tilde{A}(x)\end{pmatrix}=\mathcal{L}(x)\begin{pmatrix}A(x)\\ \tilde{A}(x)\end{pmatrix},\ \text{with}\ \mathcal{L}(x,t,\hbar)=\begin{pmatrix}\alpha(x,t,\hbar) & \beta(x,t,\hbar)\\ \gamma(x,t,\hbar) & \delta(x,t,\hbar)\end{pmatrix},$$

$\alpha,\beta,\gamma,\delta\in\mathbb{C}[[\hbar]](x)$, we obtain the quantum curve annihiliating $A(x)$:

$$\hat{y}^2-(\alpha(x)+\delta(x))\hat{y}+(\alpha(x)\delta(x)-\beta(x)\gamma(x))+\hbar\Big(\alpha'(x)-\alpha(x)\frac{\beta'(x)}{\beta(x)}-\frac{\beta'(x)}{\beta(x)}\hat{y}\Big).$$

Its classical part $\hbar\to 0$ is indeed the equation $\det(y-\mathcal{L}(x,t,0))=0$.

Gelfand–Dikii hierarchy

These systems generalize the Painlevé I equation and appear in the enumeration of maps in the large size limit. For all genus 0 spectral curves with $y^2 = P_{\mathrm{odd}}$, P_{odd} an odd polynomial of x, the associated isomonodromic system can be written as a Gelfand–Dikii system.

- Gelfand–Dikii polynomials recursively defined as differential polynomials of a function $U(t)$.

Gelfand–Dikii hierarchy

These systems generalize the Painlevé I equation and appear in the enumeration of maps in the large size limit. For all genus 0 spectral curves with $y^2 = P_{\text{odd}}$, P_{odd} an odd polynomial of x, the associated isomonodromic system can be written as a Gelfand–Dikii system.

- Gelfand–Dikii polynomials recursively defined as differential polynomials of a function $U(t)$.

Associated Lax pair given by

$$\mathcal{R}(x,t,\hbar) = \begin{pmatrix} 0 & 1 \\ x + 2U(t) & 0 \end{pmatrix}, \quad \mathcal{L}(x,t,\hbar) = \sum_{j=0}^{m} \tilde{t}_j \mathcal{L}_j(x,t,\hbar),$$

with zero curvature equation:

$$\hbar \frac{\partial}{\partial t}\mathcal{L}(x,t,\hbar) + \hbar \frac{\partial}{\partial x}\mathcal{R}(x,t,\hbar) = [\mathcal{R}(x,t,\hbar), \mathcal{L}(x,t,\hbar)].$$

The spectral curve, in the limit $\hbar \to 0$: $\det(y - \mathcal{L}(x,t,0)) = 0$ is always a genus 0 curve.

It admits the rational parametrization

$$\begin{cases} x(z) = z^2 - 2u(t) \\ y(z) = \sum_{j=0}^m \tilde{t}_j \left(z^{2j+1}(1-2u(t)/z^2)^{j+\frac{1}{2}} \right)_+ \end{cases},$$

$()_+ \rightsquigarrow$ the positive part in the Laurent series expansion near $z = \infty$:

$$y(z) = \sum_{j=0}^{m} \tilde{t}_j \sum_{k=0}^{j} (-u)^k \frac{(2j+1)!!}{(2j-2k+1)!!} \, z^{2j-2k+1}.$$

$\Psi(x,t,\hbar) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}$ WKB $\hbar$ formal series solution of the Lax system:

$$\hbar \frac{\partial}{\partial x} \Psi(x,t,\hbar) = \mathcal{L}(x,t,\hbar)\Psi(x,t,\hbar) \,,\; \hbar \frac{\partial}{\partial t} \Psi(x,t,\hbar) = \mathcal{R}(x,t,\hbar)\Psi(x,t,\hbar).$$

It admits the rational parametrization

$$\begin{cases} x(z) = z^2 - 2u(t) \\ y(z) = \sum_{j=0}^{m} \tilde{t}_j \left(z^{2j+1} (1 - 2u(t)/z^2)^{j+\frac{1}{2}} \right)_+ \end{cases},$$

$()_+ \rightsquigarrow$ the positive part in the Laurent series expansion near $z = \infty$:

$$y(z) = \sum_{j=0}^{m} \tilde{t}_j \sum_{k=0}^{j} (-u)^k \frac{(2j+1)!!}{(2j-2k+1)!!} \, z^{2j-2k+1}.$$

$\Psi(x,t,\hbar) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}$ WKB $\hbar$ formal series solution of the Lax system:

$$\hbar \frac{\partial}{\partial x} \Psi(x,t,\hbar) = \mathcal{L}(x,t,\hbar)\Psi(x,t,\hbar) \; , \; \hbar \frac{\partial}{\partial t} \Psi(x,t,\hbar) = \mathcal{R}(x,t,\hbar)\Psi(x,t,\hbar).$$

We showed that the formal series $\psi([z]-[\infty],t,\hbar)$ coincides with

$$A(x,t,\hbar) = \frac{1}{\sqrt{2z}} e^{\hbar^{-1} \int_0^z y dx} e^{\sum_{(g,n)\neq(0,1),(0,2)} \frac{\hbar^{2g-2+n}}{n!} \int_\infty^z \cdots \int_\infty^z \omega_{g,n}},$$

and is annihiliated by the quantum curve

$$\hat{y}^2 - (\alpha(x)+\delta(x))\hat{y} + (\alpha(x)\delta(x) - \beta(x)\gamma(x)) + \hbar \left(\alpha'(x) - \alpha(x) \frac{\beta'(x)}{\beta(x)} - \frac{\beta'(x)}{\beta(x)} \hat{y} \right).$$

Outline

1 Quantum curves and topological recursion

2 Spectral curves

3 Loop equations and deformation parameters

4 Partition function, wave functions and PDEs

5 Non-perturbative corrections and quantum curve
- Isomonodromic deformations

6 Questions and future work

Quantizing any rank d spectral curve

(work in progress with B. Eynard, O. Marchal and N. Orantin)

- Fix N distinct points $\Lambda_i \in \mathbb{C}P^1$ and a Riemann surface Σ of genus $\hat{g}$. Let $x : \Sigma \to \mathbb{C}P^1$ be a d-sheeted ramified covering. The poles of $\omega_{0,1}$ are prescribed to be at ζ for which $x(\zeta) = \infty$ and at $x^{-1}(\Lambda_i)$, $\forall i = 1, \ldots, N$. Our spectral curve is of the form:

$$P(x,y) = \sum_{k=0}^{d} y^{d-k}(-1)^k P_k(x).$$

We assume the spectral curve to have simple branchpoints away from the poles Λ_i, $i = 1, \ldots, N$.

Quantizing any rank d spectral curve

(work in progress with B. Eynard, O. Marchal and N. Orantin)

- Fix N distinct points $\Lambda_i \in \mathbb{C}P^1$ and a Riemann surface Σ of genus $\hat{g}$. Let $x : \Sigma \to \mathbb{C}P^1$ be a d-sheeted ramified covering. The poles of $\omega_{0,1}$ are prescribed to be at ζ for which $x(\zeta) = \infty$ and at $x^{-1}(\Lambda_i)$, $\forall i = 1, \ldots, N$. Our spectral curve is of the form:

$$P(x, y) = \sum_{k=0}^{d} y^{d-k} (-1)^k P_k(x).$$

  We assume the spectral curve to have simple branchpoints away from the poles Λ_i, $i = 1, \ldots, N$.
- Higher rank loop equations and variational formulas with respect to parameters of the spectral curve.

Quantizing any rank d spectral curve

(work in progress with B. Eynard, O. Marchal and N. Orantin)

- Fix N distinct points $\Lambda_i \in \mathbb{C}P^1$ and a Riemann surface Σ of genus $\hat{g}$. Let $x : \Sigma \to \mathbb{C}P^1$ be a d-sheeted ramified covering. The poles of $\omega_{0,1}$ are prescribed to be at ζ for which $x(\zeta) = \infty$ and at $x^{-1}(\Lambda_i)$, $\forall i = 1, \ldots, N$. Our spectral curve is of the form:

$$P(x,y) = \sum_{k=0}^{d} y^{d-k}(-1)^k P_k(x).$$

  We assume the spectral curve to have simple branchpoints away from the poles Λ_i, $i = 1, \ldots, N$.
- Higher rank loop equations and variational formulas with respect to parameters of the spectral curve.
- Non-perturbative PDEs anihilating wave-functions over any degree 0 divisor (as quantum curves). Further exploration and understanding of these types of quantum curves.

Quantizing any rank d spectral curve

(work in progress with B. Eynard, O. Marchal and N. Orantin)

- Fix N distinct points $\Lambda_i \in \mathbb{C}P^1$ and a Riemann surface Σ of genus $\hat{g}$. Let $x : \Sigma \to \mathbb{C}P^1$ be a d-sheeted ramified covering. The poles of $\omega_{0,1}$ are prescribed to be at ζ for which $x(\zeta) = \infty$ and at $x^{-1}(\Lambda_i)$, $\forall i = 1, \ldots, N$. Our spectral curve is of the form:

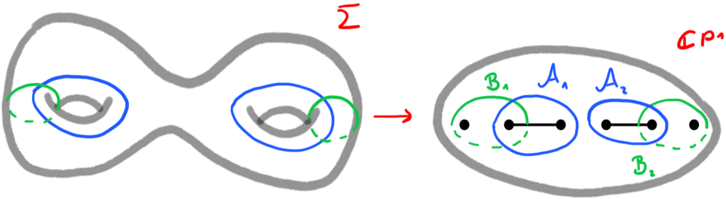
$$P(x,y) = \sum_{k=0}^{d} y^{d-k}(-1)^k P_k(x).$$

  We assume the spectral curve to have simple branchpoints away from the poles Λ_i, $i = 1, \ldots, N$.
- Higher rank loop equations and variational formulas with respect to parameters of the spectral curve.
- Non-perturbative PDEs anihilating wave-functions over any degree 0 divisor (as quantum curves). Further exploration and understanding of these types of quantum curves.
- Relation to method sketched in Eynard, '17.
- Relation to isomonodromic systems? Can we generalize the method of Marchal–Orantin, '19?

Questions and remarks

- From TR, relation to intersection theory granted. Interesting enumerative geometry in higher genus TR problems?
- Further explore the connection with summability, transseries, exact WKB, resurgence.
- Relation to the topological type property approach (can that be proved for higher genus spectral curves?).
- Generalization to spectral curves allowing higher ramifications (to apply this method, we would first have to generalize the variational formulas).
- Extend the result to a ramified covering of surfaces other than $\mathbb{C}P^1$.
- Generalization to difference equations? (Subtleties including K_2 condition of Gukov–Sułkowski '12?).
- General relation between Virasoro constraints (or even Kontsevich–Soibelman '17, ABCD of Andersen–Borot–Chekhov–Orantin '17) and quantum curves.

Merci de votre attention !



Tak for din opmærksomhed!