Quantum curves, isomonodormic systems and Riemann-Hilbert

Quantizing hyper-elliptic curves

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Elba's talk: Quantization of hyperelliptic curve

$$y^2 = \phi(x) \mapsto \left(\hbar \frac{\partial}{\partial x}\right)^2 \psi(x,\hbar) = \left[\hbar^2 R(x) \frac{\partial}{\partial x} + \hbar Q(x) + \mathcal{H}(x,\hbar)\right] \psi(x,\hbar)$$

such that

$$\lim_{\hbar \to 0} \mathcal{H}(x,\hbar) = \phi(x).$$

Today's talk: study of the corresponding connection

$$\hbar \frac{\partial}{\partial x} \Psi(x,\hbar) = L(x,\hbar)\Psi(x,\hbar)$$

1. Goal 1 : understand $R(x)$, $Q(x)$ and $\mathcal{H}(x)$ from the perspective of moduli space of connection (and its symplpectic structure)
2. Goal 2 : describe the monodromies and Stokes matrices of $\Psi(x,\hbar)$
3. Goal 3 : Summarize some of the numerous open questions

Topological recursion

Input:

- A rational function $\phi(x)$ with poles at $x = X_j$ and $x = \infty$ defining a genus g Riemann surface Σ by $y^2 = \phi(x)$;
- A basis of cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ on Σ ($\Leftrightarrow$ a choice of polarization for the quantization);

Output: Differential forms

$$\omega_{0,1} := ydx,$$

$\omega_{0,2}$ is a Bergman kernel with vanishing $\mathcal{A}$-periods

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2} + \text{holomorphic as } z_1 \to z_2$$

and

$$\forall 2h - 2 + n \geq 1\,,\ \omega_{h,n} \in H^0(\Sigma^n, (K_\Sigma(\mathcal{R}))^{\boxtimes n})$$

Perturbative wave function

$$\psi_\pm^{pert}(x) := \exp\left[\sum_{h,n} \frac{\hbar^{2h-2+n}}{n!} \overbrace{\int_{\infty_\pm}^{z(x)^\pm} \cdot \int_{\infty_\pm}^{z(x)^\pm}}^{n} \omega_{h,n}\right]$$

The moduli of the spectral curve can be described by the coefficients of the decomposition

$$\phi(x) = \sum_{k=0}^{2(r_\infty - 2)} H_{\infty,k} x^k + \sum_{\nu=1}^{n} \sum_{k=1}^{2r_\nu} \frac{H_{\nu,k}}{(x - X_\nu)^k} \tag{1}$$

We prefer working with the following moduli.

Moduli

- Moduli at poles

$$\omega_{0,1}[\phi] = \pm \sum_{k=1}^{r_\nu} T_{\nu,k} \frac{dx}{(x - X_\nu)^k} + O\left(dx\right),$$

$$\omega_{0,1}[\phi] = \pm \sum_{k=1}^{r_\infty} T_{\infty,k} (x^{-1})^{-k} d(x^{-1}) + O(d(x^{-1}))$$

- Periods

$$\epsilon_i := \oint_{\mathcal{A}_i} \omega_{0,1}$$

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- Periods

$$\epsilon_i := \oint_{\mathcal{A}_i} \omega_{0,1}$$

The coefficients of the partial fraction decomposition can be decomposed in "Casimirs" and "Hamiltonians" (cf. integrable system later in this talk)

$$\phi(x) = \sum_{k=r_\infty-3}^{2(r_\infty-2)-n_\infty} H_{\infty,k}(\mathbf{T})\, x^k + \sum_{k=0}^{r_\infty-4} H_{\infty,k}(\mathbf{T},\epsilon)\, x^k + \sum_{\nu=1}^{n} \left(\sum_{k=r_\nu+1}^{2r_\nu} \frac{H_{\nu,k}(\mathbf{T})}{(x - X_\nu)^k} + \sum_{k=1}^{r_\nu} \frac{H_{\nu,k}(\mathbf{T},\epsilon)}{(x - X_\nu)^k} \right).$$

Theorem [Marchal-O. , Eynard-Garcia-Failde]

The perturbative wave function is solution to the PDE

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial}{\partial T_{\infty,k}} - \hbar^2 \sum_{\nu=1}^{n} \sum_{k \in K_{b_\nu}} U_{\nu,k}(x) \frac{\partial}{\partial T_{\nu,k}} - \phi(x)\right] \psi_\pm(x,\hbar) = 0$$

with

- $K_\infty = [\![2, r_\infty - 2]\!]$ and $\forall\, k \in K_\infty$:

$$U_{\infty,k}(x) := (k-1) \sum_{l=k+2}^{r_\infty} T_{\infty,l}\, x^{l-k-2}$$

- $K_\nu = [\![2, r_\nu + 1]\!]$ and $\forall\, k \in K_\nu$:

$$U_{\nu,k}(x) := (k-1) \sum_{l=k-1}^{r_\nu} T_{\nu,l}\, (x - X_\nu)^{-l+k-2}$$

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Perturbative monodromies

The perturbative wave functions $\psi_\pm$ satisfy the following properties.

- For $i \in [\![1,g]\!]$, the function $\psi_\pm(x,\hbar,\mathbf{T},\epsilon)$ has a formal monodromy along $x(\mathcal{A}_i)$ given by

$$\psi_\pm(x,\hbar,\mathbf{T},\epsilon) \mapsto e^{\pm 2\pi i \frac{\epsilon_i}{\hbar}} \psi_\pm(x,\hbar,\mathbf{T},\epsilon). \tag{1}$$

- For $i \in [\![1,g]\!]$, the function $\psi_\pm(x,\hbar,\mathbf{T},\epsilon)$ has a formal monodromy along $x(\mathcal{B}_i)$ given by

$$\psi_\pm(x,\hbar,\mathbf{T},\epsilon) \mapsto \psi_\pm(x,\hbar,\mathbf{T},\epsilon \pm \hbar\,\mathbf{e}_i) = e^{\pm 2\pi i \hbar \frac{\partial}{\partial \epsilon_i}} \psi_\pm(x,\hbar,\mathbf{T},\epsilon) \tag{2}$$

  where $\mathbf{e}_i \in \mathbb{C}^g$ is the vector with the i^{th} component equal to 1 and all others vanishing.

$\Rightarrow \psi_\pm$ have non-trivial monodromies on the base curve $\mathbb{P}^1 \setminus x(\mathcal{R})$.

Non-perturbative wave function

Let us define the Fourier transforms

$$\Psi_{\pm}(x,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) := \sum_{\mathbf{k}\in\mathbb{Z}^g} e^{\frac{2\pi i}{\hbar}\sum_{j=1}^{g} k_j\rho_j}\, \psi_{\pm}(x,\hbar,\mathbf{T},\boldsymbol{\epsilon}+\hbar\mathbf{k}).$$

They satisfy the same PDE

$$\left[\hbar^2\frac{\partial^2}{\partial x^2} - \hbar^2\sum_{k\in K_\infty} U_{\infty,k}(x)\frac{\partial}{\partial T_{\infty,k}} - \hbar^2\sum_{\nu=1}^{n}\sum_{k\in K_{b_\nu}} U_{\nu,k}(x)\frac{\partial}{\partial T_{\nu,k}} - \phi(x)\right]\Psi_{\pm}(x,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) = 0$$

and have the monodromies

$$\Psi_{\pm}(x+x(\mathcal{A}_i),\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) \mapsto e^{\pm 2\pi i\frac{\epsilon_j}{\hbar}}\Psi_{\pm}(x,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}).$$

$$\Psi_{\pm}(x+x(\mathcal{B}_i),\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) \mapsto e^{\mp 2\pi i\frac{\rho_j}{\hbar}}\Psi_{\pm}(x,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}).$$

$\Rightarrow$ The non-perturbative wave functions have good monodromies on base curve $\mathbb{P}^1 \setminus x(\mathcal{R})$.

"Change of basis"

Let us define

$$\forall p \in \{\infty\} \cup [\![1,n]\!]\,,\ \forall k \in K_p\ :\ \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar \frac{\partial \Psi_+}{\partial T_{p.k}} & \hbar \frac{\partial \Psi_-}{\partial T_{p.k}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Q_{p,k} & R_{p,k} \end{pmatrix} \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar \frac{\partial \Psi_+}{\partial x} & \hbar \frac{\partial \Psi_-}{\partial x} \end{pmatrix}$$

in such a way that

$$\hbar \frac{\partial \Psi_\pm(x)}{\partial T_{p.k}} = Q_{p,k}(x)\,\Psi_\pm(x) + R_{p,k}(x)\,\hbar \frac{\partial \Psi_\pm(x)}{\partial x}$$

Monodromies $\Rightarrow$ $Q_{p,k}$ and $R_{p,k}$ are rational functions of x on the base curve.
They might have poles at $x \in \{\infty, X_\nu, x(\mathcal{R})\}$ and at the zeroes of the Wronskian

$$W(x) := \hbar \left(\frac{\partial \Psi_+}{\partial x} \Psi_- - \Psi_+ \frac{\partial \Psi_-}{\partial x} \right) = \kappa \frac{\prod_{i=1}^{g} (x - q_i(\hbar))}{\prod_\nu (x - X_\nu)^{r_\nu}}.$$

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in such a way that

$$\hbar \frac{\partial \Psi_\pm(x)}{\partial T_{p.k}} = Q_{p,k}(x)\, \Psi_\pm(x) + R_{p,k}(x)\, \hbar \frac{\partial \Psi_\pm(x)}{\partial x}$$

From PDE to a quantum curve

One has

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 R(x) \frac{\partial}{\partial x} - \hbar Q(x) - \phi(x)\right] \Psi_\pm = 0$$

where

$$R(x) := \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) R_{p,k}(x) \qquad \text{and} \qquad Q(x) := \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) Q_{p,k}(x).$$

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Theorem

- $R(x)$ is the logarithmic derivative of the Wronskian

$$R(x) = \frac{\partial \log W(x)}{\partial x}.$$

- Compatibility between $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial T_{k,\nu}}$ $\Rightarrow$ $R(x)$ and $Q(x)$ do not have any pole when $x \in x(\mathcal{R})$.

Definition

For any Lie algebra $\mathfrak{g}$ together with a loop $\mathcal{C}$ on $\mathbb{P}^1$, one defines a loop algebra $\tilde{\mathfrak{g}}$ as the space

$$\tilde{\mathfrak{g}} := \{\text{smooth maps } L : \mathcal{C} \to \mathfrak{g}\}$$

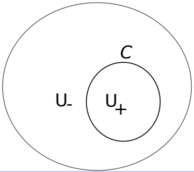
together with a polarization

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_+ \oplus \tilde{\mathfrak{g}}_-$$

where $\tilde{\mathfrak{g}}_+ := \{L \in \tilde{\mathfrak{g}} | L \text{admits a holomorphic extension to } U_+\}$.
One can define an Ad-invariant inner product $< \cdot, \cdot >: \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \mathbb{C}$ by

$$\forall (L_1, L_2) \in \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}\, , \; \langle L_1, L_2 \rangle := \frac{1}{2\pi i} \oint_{x \in \mathcal{C}} \mathrm{Tr}\, [L_1(x) \cdot L_2(x)]\, dx.$$

This allows the identification of $\tilde{\mathfrak{g}}$ with its dual $\tilde{\mathfrak{g}}^*$ in such a way that the $\tilde{\mathfrak{g}}^*_\pm$ can be identified with $\tilde{\mathfrak{g}}_\mp$.



We consider $\mathcal{C}$ as a small contour encircling $x = \infty$.
This allows to identify $\tilde{\mathfrak{g}}^+$ (resp. $\tilde{\mathfrak{g}}^-$) with elements of $\mathfrak{g}[[x]]$ (resp. $x^{-1}\mathfrak{g}[[x^{-1}]]$).

The exponentiated group $\tilde{G}^*$ acts by coadjoint action through

$$\forall (f,g) \in \tilde{\mathfrak{g}}^* \times \tilde{\mathfrak{g}}\,,\ \forall X \in \tilde{G}^*\,,\ \mathrm{Ad}^*_X(f)(g) = \frac{1}{2\pi i} \oint_{x\in\mathcal{C}} \mathrm{Tr}\,([X,f]g)\,.$$

Classical R-matrix

The classical R-matrix construction, defines the bracket

$$\forall (L_1, L_2) \in \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}},\ [L_1, L_2]_R := [R(L_1), L_2] + [L_1, R(L_2)]$$

where

$$R := \frac{1}{2}(P_+ - P_-)$$

with $P_\pm$ being the projection operator to $U_\pm$. This defines a Lie-Poisson structure on $\tilde{\mathfrak{g}}^*$ through the bracket

$$\forall (f,g) \in \tilde{\mathfrak{g}}^* \times \tilde{\mathfrak{g}}^*\,,\ \{f,g\}_R(\mu) := \langle \mu, [df(\mu), dg(\mu)]_R \rangle\,.$$

AKS theorem

Let us denote by $\mathcal{I}$ the set of spectral invariants, i.e. the set of Ad^*-invariant polynomials on $\tilde{\mathfrak{g}}^*$. Then the elements of $\mathcal{I}$ Poisson commute. For $H \in \mathcal{I}$, one has Hamilton's equations

$$\frac{dL}{dt} = [P_\sigma(dH), L] \qquad \text{where} \qquad P_\sigma := \frac{1}{2}\left[(1+\sigma)P_+ + (\sigma - 1)P_-\right].$$

For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathcal{I}$ is generated by

$$\forall l \in \mathbb{Z}\,,\; h_l := \operatorname*{Res}_{x\to\infty} x^{-l-1}\,\mathrm{Tr}\, L(x) dx = \left\langle L(x), x^{-l-1} \right\rangle .$$

Definition

Let us define the finite dimensional subspace

$$\hat{\mathfrak{g}}^* := \left\{ L(x) := \sum_{i=0}^{r_0} L_{0,i} x^i + \sum_{\nu=1}^{n} \sum_{i=1}^{r_\nu} \frac{L_{\nu,i}}{(x - X_\nu)^i} \,,\, (L_{\nu,k}) \in \mathfrak{g}^r \right\}$$

Spectral invariants and Hamilton's equations

One has generators of the set of spectral invariants $\hat{\mathcal{I}}$ given by the coefficients $H_{\nu,i}$ of

$$\mathrm{Tr}\, \left(L(x)^2\right) = \sum_{i=0}^{2\, r_0} H_{0,i} x^i + \sum_{\nu=1}^{n} \sum_{i=1}^{2r_\nu} \frac{H_{\nu,i}}{(x - X_\nu)^i}.$$

The associated Hamilton's equations read

$$\frac{dL(x)}{dt_{\nu,i}} = \left[A_{\nu,i}, L(x)\right] \qquad \text{where} \qquad A_{\nu,i} = 2\left[(x - X_\nu)^{i-1}\, L(x)\right]_{-,X_\nu} .$$

Spectral invariants and Hamilton's equations

$$\mathrm{Tr}\ \left(L(x)^2\right) = \sum_{i=0}^{2\,r_0} H_{0,i} x^i + \sum_{\nu=1}^{n} \sum_{i=1}^{2r_\nu} \frac{H_{\nu,i}}{(x - X_\nu)^i}.$$

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This leads to iso-spectral deformations $\frac{\partial \det(y - L(x))}{\partial t_{\nu,l}} = 0$.

Casimirs and Hamiltonians

For any $\nu \neq 0$, $H_{\nu,i}$ is a Casimir for $r_\nu + 1 \leq i \leq 2r_\nu$ while $H_{0,i}$ is a Casimir for $r_0 \leq i \leq 2r_0$. One can check that the number of non-Casimir Hamiltonians is equal to

$$r := r_0 + \sum_{\nu=1}^{n} r_\nu$$

which gives half the dimension of a generic symplectic leaf.

Reduction and choice of gauge

Fixing the value of H_{0,r_0-1} leads to a symplectic reduction by modding out by the elements of the stabilizer of L_{0,r_0}. The resulting reduced space has dimension $2d-2=2g$ where g is the genus of the spectral curve

$$\det(y - L(x)) = 0.$$

By conjugation with $Stab_{L_{0,r_0}} = Stab_{\sigma_3}$, we can choose to fix one element (typically L_{0,r_0-1}) of the form

$$L_{\nu,k} = \begin{pmatrix} U & V \\ 1 & -U \end{pmatrix}$$

so that

$$L(x) = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix}$$

with

$$P(x) = \frac{Pol_{g+1}(x)}{\prod_{\nu=1}^{n}(x-X_\nu)^{r_\nu}} \quad , \quad W(x) = \frac{Pol_g(x)}{\prod_{\nu=1}^{n}(x-X_\nu)^{r_\nu}} \quad \text{and} \quad M(x) = \frac{Pol_g(x)}{\prod_{\nu=1}^{n}(x-X_\nu)^{r_\nu}}.$$

Choice of gauge

$$L(x) = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix}$$

with

$$P(x) = \frac{Pol_{g+1}(x)}{\prod_{\nu=1}^{n}(x - X_\nu)^{r_\nu}} \quad , \quad W(x) = \frac{\prod_{i=1}^{g}(x - q_i)}{\prod_{\nu=1}^{n}(x - X_\nu)^{r_\nu}} \quad \text{and} \quad M(x) = \frac{Pol_g(x)}{\prod_{\nu=1}^{n}(x - X_\nu)^{r_\nu}}.$$

Spectral Darboux coordinates [Harnad et al.]

One has a set of Darboux coordinates $(q_i, p_i)_{i=1}^{g}$ given by

$$W(q_i) = 0 \qquad \text{and} \qquad p_i := P(q_i).$$

They satisfy

$$\det(p_i - L(q_i)) = 0$$

and

$$\frac{\partial q_i}{\partial t_{\nu,l}} = \frac{\partial H_{\nu,l}}{\partial p_i} \qquad \text{and} \qquad \frac{\partial p_i}{\partial t_{\nu,l}} = -\frac{\partial H_{\nu,l}}{\partial q_i}.$$

Autonomous system

Up to now, one has an autonomous system: $H_{\nu,l}$ does not depend on $t_{\mu,j}$ and

$$\frac{dL(x)}{dt_{\nu,i}} = \left[A_{\nu,i}, L(x)\right].$$

Non-Autonous system

Let us now assume that $L(x) = L(x,a)|_{a=t_{\nu,l}}$ depends explicitly on $t_{\nu,l}$ in such a way that

$$\left.\frac{\partial L(x,a)}{\partial a}\right|_{a=t_{\nu,l}} = \frac{\partial A_{\nu,l}}{\partial x}.$$

Hamilton's equations now include this explicit dependence and read

$$\frac{dL(x)}{dt_{\nu,l}} - \frac{\partial A_{\nu,l}}{\partial x} = \left[A_{\nu,l}, L(x)\right]$$

which is the compatibility condition for the isomonodromic system

$$\begin{cases} \frac{\partial}{\partial x}\Psi(x,t) = L(x,t)\Psi(x,t) \\ \frac{\partial}{\partial t_{\nu,l}}\Psi(x,t) = A_{\nu,l}(x,t)\Psi(x,t) \end{cases}.$$

Painlevé 2 isospectral system

Let us consider again $n = 0$ and $r_0 = 2$ with $L_{0,2} = \sigma_3$. This implies that the characteristic polynomial of $L(x)$ is a degree 4 polynomial in x and the non-Casimir Hamiltonian are given again by

$$H_{0,0} = \operatorname*{Res}_{x\to\infty} x^{-1}\,\mathrm{Tr}\,\left[L(x)\right]^2\,dx \qquad \text{and} \qquad H_{0,1} = \operatorname*{Res}_{x\to\infty} x^{-2}\,\mathrm{Tr}\,\left[L(x)\right]^2\,dx$$

with the associated auxiliary matrices

$$\frac{A_{0,0}}{2} = \left[x^{-1}L(x)\right]_+ = \sigma_3\,x + L_{0,1} \qquad \text{and} \qquad \frac{A_{0,1}}{2} = \left[x^{-2}L(x)\right]_+ = \sigma_3.$$

For simplicity let us consider a symplectic leaf of the form $(H_{0,3}, H_{0,2}, H_{0,1}) = (0, \alpha_2, \alpha_1)$.

Reduced system and Darboux coordinates

Considering a representative of the reduced orbit as before, one has a Lax matrix of the form

$$L(x) = \sigma_3 x^2 + \begin{pmatrix} 0 & v_1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} u_0 & v_0 \\ w_0 & -u_0 \end{pmatrix}$$

with $2u_0 + v_1 = \alpha_2$ and $v_0 + v_1 w_0 = \alpha_1$. One obtains the spectral Darboux coordinates

$$\begin{cases} q = -w_0 \\ p = q^2 + u_0 \end{cases}$$

Remark that $\tilde{p} = p - q^2$ gives an alternative Darboux coordinate dual to q. One gets

$$L(x) = \sigma_3 x^2 + \begin{pmatrix} 0 & \alpha_2 - 2\tilde{p} \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} \tilde{p} & \alpha_1 + q\left[\alpha_2 - 2\tilde{p}\right] \\ -q & -\tilde{p} \end{pmatrix}$$

$$A_{0,0}(x) = 2\sigma_3 x + 2 \begin{pmatrix} 0 & \alpha_2 - 2\tilde{p} \\ 1 & 0 \end{pmatrix}$$

From isospectral to isomonodromic

Following the general procedure, one can identify the isomonodromic time with $t_{0,0}$ and consider

$$\tilde{L}(x,t) := \sigma_3 x^2 + L_{0,1} x + L_{0,0} + 2t\sigma_3$$

leading to the isomonodromic system

$$\begin{cases} \frac{\partial}{\partial x}\Psi = \left[\sigma_3 x^2 + L_{0,1} x + L_{0,0} + 2t_{0,0}\sigma_3\right]\Psi \\ \frac{\partial}{\partial t_{0,0}}\Psi = \left[\sigma_3\, x + \beta\right]\Psi \end{cases} .$$

The time evolution of q recovers Painlevé 2 equation.

Isomonodromic system

Linear system associated to the quantum curve

We have defined the non-perturbative partition functions $\Psi_\pm(x)$ which are solutions to our quantum curve equation.

Lax representation

For any rational function $P(x)$, let us define an associated

$$\hat{\Psi}_\pm(x) := \frac{1}{W(x)}\left[P(x)\Psi_\pm(x) + \hbar\frac{\partial \Psi_\pm(x)}{\partial x}\right].$$

Then

$$\hbar\frac{\partial}{\partial x}\begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix}\begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix}$$

where

$$W(x) = \kappa\frac{\prod_{i=1}^{g}(x-q_i)}{\prod_{\nu=1}^{n}(x-X_\nu)^{r_\nu}} \quad , \quad M(x) = \frac{\hbar\frac{\partial P(x)}{\partial x} - \hbar\frac{\partial \log W(x)}{\partial x}P(x) - P(x)^2 + \hbar Q(x) + \phi(x)}{W(x)}.$$

and

$$\forall i = 1,\dots,g\,,\ P(q_i) = p_i$$

with

$$\forall j \in [\![1,g]\!]\ :\ p_j := -\hbar\,\frac{\partial \log \Psi_+(x)}{\partial x}\Big|_{x=q_j} = -\hbar\,\frac{\partial \log \Psi_-(x)}{\partial x}\Big|_{x=q_j}.$$

Lax representation-choice of Gauge

We choose

$$P(x) = \frac{T_{\infty,r_\infty} x^{g+1} + \left(T_{\infty,r_\infty} + \frac{\hbar}{2}\right) x^g + \sum_{k=0}^{g-1} a_k x^k}{\prod_{\nu=1}^n (x - X_\nu)^{r_\nu}}$$

defined by the condition

$$\forall i = 1, \dots, g\,,\ P(q_i) = p_i$$

Then

$$\hbar \frac{\partial}{\partial x} \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix} \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix}$$

where

$$W(x) = \kappa \frac{\prod_{i=1}^g (x - q_i)}{\prod_{\nu=1}^n (x - X_\nu)^{r_\nu}}$$

and $M(x) = \frac{Pol_g(x)}{\prod_{\nu=1}^n (x-X_\nu)^{r_\nu}}$.

$\Rightarrow$ We have built explicitly a point in the moduli space of connections described above! This means in particular that q_i and p_i satisfy the evolution equations associated to this system.

- From the non-perturbative completion, we have built a (family of) connection $d - \frac{L(x,\hbar)}{\hbar} dx$ on the base curve with free parameters $\boldsymbol{\epsilon}$, $\boldsymbol{\rho}$.
- **Conjecture**: there exist values of $\boldsymbol{\epsilon}$, $\boldsymbol{\rho}$ making the trans-series involved summable (Boutroux condition + quantization condition?).
- For any isomonodormic system, the connection $\sum_{\nu,l} H_{\nu,l} dt_{\nu,l}$ is flat and one can define isomonodromic tau functions by integration through

$$\frac{\partial \ln \tau}{\partial t_{\nu,l}} = H_{\nu,l}.$$

- **Conjecture**: The non-perturbative partition function

$$Z^{NP}(\hbar, \boldsymbol{\epsilon}, \boldsymbol{\rho}) := \sum_{\mathbf{k} \in \mathbb{Z}^g} e^{\frac{2\pi i}{\hbar} \sum_{j=1}^{g} k_j \rho_j} Z(\hbar, \boldsymbol{\epsilon} + \hbar \mathbf{k})$$

where

$$Z(\hbar, \boldsymbol{\epsilon} + \hbar \mathbf{k}) := \exp(\sum_{g=0}^{\infty} \hbar^{2g-2} \omega_{g,0})$$

is a tau function

$$\frac{\partial \ln Z^{NP}(\hbar, \boldsymbol{\epsilon}, \boldsymbol{\rho})}{\partial t_{\nu,l}} = H_{\nu,l}.$$

- From the non-perturbative completion, we have built a (family of) connection $d - \frac{L(x,\hbar)}{\hbar} dx$ on the base curve with free parameters ϵ, $\boldsymbol{\rho}$.
- **Conjecture**: there exist values of ϵ, $\boldsymbol{\rho}$ making the trans-series involved summable (Boutroux condition + quantization condition?).
- For any isomonodormic system, the connection $\sum_{\nu,l} H_{\nu,l} dt_{\nu,l}$ is flat and one can define isomonodromic tau functions by integration through

$$\frac{\partial \ln \tau}{\partial t_{\nu,l}} = H_{\nu,l}.$$

- **Conjecture**: The non-perturbative partition function is a tau function
- Can we have a Fredholm determinant representation of the non-perturbative partition function? (cf. work of Cafasso, Lisovyy, Teschner, Mariño...)
- How does this connection depend on the choice of cycles $(\mathcal{A}_i, \mathcal{B}_i)$? (cf. second part of this talk)
- Can we generalise it for other Lie algebras and base curves?
- Is there a more geometric picture hiding behind these computations?

Initial data from the base curve

Starting from the base curve and not the spectral curve

We started from the data of

- a quadratic differential $\phi(dx)^2$ on $\mathbb{P}^1$,
- a basis of cycles on the cover defined by this quadratic differential.

We could get this second type of initial data from quantities purely on the base curve $\mathbb{P}^1$ (which makes sense from the integrable system/moduli space of connections point of view).

One can replace the choice of cycle on the cover by a choice of Stokes curves/spectral network on the base curve.

From now on, we consider the Painlevé 1 example following [Iwaki].

Painlevé 1 example

Quantization [Iwaki]

Let us start from a polynomial

$$\phi(x) = 4x^3 + 2tx + H(t,\epsilon) = 4(x-e_A)(x-e_B)(x-e_{AB}).$$

The quantization procedure gives rise to the compatible system

$$\hbar \frac{\partial}{\partial x}\Psi(x,\hbar) = \begin{pmatrix} p & x^2+qx+q^2+\frac{t}{2} \\ 4(x-q) & -p \end{pmatrix} \Psi(x,\hbar) \quad , \quad \hbar\frac{\partial}{\partial t}\Psi(x,\hbar) = \begin{pmatrix} 0 & x+\frac{q}{2} \\ 2 & 0 \end{pmatrix}\Psi(x,\hbar)$$

with

$$q = -\hbar^2 \frac{\partial^2 \log Z^{NP}(\hbar,\epsilon,\rho)}{\partial t^2}$$

which is solution to Painlevé 1 equation

$$\hbar^2 \frac{\partial^2 q}{\partial t^2} = 6q^2 + t$$

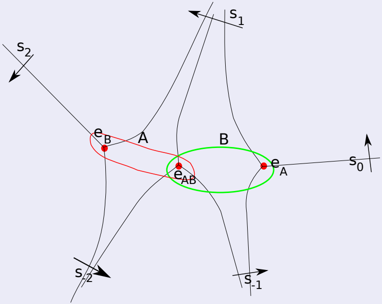
This is valid for any choice of cycles $(\mathcal{A},\mathcal{B})$.

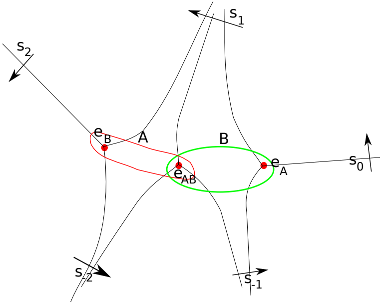
Stokes graph

The Stokes curves are defined by the set of points x such that

$$\mathrm{Im} \int_e^x \phi(x)^{\frac{1}{2}}\, dx = 0$$

for some $e \in \{e_A, e_B, e_{AB}\}$.
There exist values of (t, H) such that the Boutroux condition
$\mathrm{Re} \oint_\gamma \phi(x)^{\frac{1}{2}}\, dx = 0$ for any closed curve γ and the Stokes graph takes the form

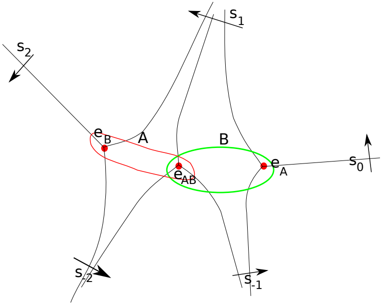




We can choose the basis of cycles $(\mathcal{A}, \mathcal{B})$ as pre-images of closed curves encircling critical values. For this choice of cycles, one can compute the Stokes matrices of our solution around infinity. One has

$$\begin{pmatrix} 1 & s_{-2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & i \cdot \left(e^{\frac{-2\pi i \rho}{\hbar}} - e^{\frac{2\pi i(\epsilon-\rho)}{\hbar}} \right) \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ s_{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i \cdot \left(e^{\frac{-2\pi i \epsilon}{\hbar}} - e^{\frac{-2\pi i(\epsilon-\rho)}{\hbar}} \right) & 1 \end{pmatrix}$$



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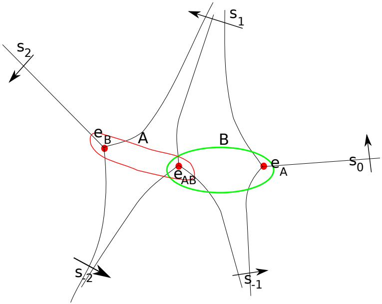
$$\begin{pmatrix} 1 & s_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & i \cdot e^{\frac{2\pi i \epsilon}{\hbar}} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i \cdot \left(e^{\frac{-2\pi i \epsilon}{\hbar}} - e^{\frac{-2\pi i(\epsilon+\rho)}{\hbar}} + e^{\frac{-2\pi i \rho}{\hbar}} \right) & 1 \end{pmatrix}$$



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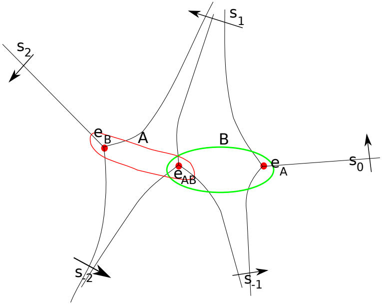
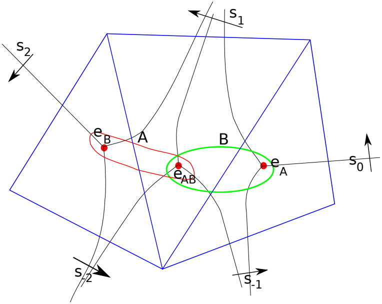


One has the cyclic relation

$$\begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & s_{-2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which give the exchange cluster relation of the A_2 cluster algebra for any ρ and ϵ.



- ρ and ϵ are coordinates on the moduli space of connections
- A choice of cycles reflects a choice on coordinates on this moduli space
- How can we change such a choice? (Stokes phenomenon when changing the argument of $\phi/\hbar$)
- In order to make this not only formal, one needs:
  1. Borel summability of the perturbative wave functions
  2. Proof of Voros connection formula for the WKB series
  3. Summability of the trans-series

  This probably requires both a quantization condition and a Boutroux curve condition.
- How is it linked to Marcos' quantization condition?