TS/ST correspondence via (non-autonomous) Toda equations

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Preface

In the past few decades, lots of new results have been obtained exploiting string dualities which connect very different theories and their math aspects.

- AdS/CFT
- Mirror symmetry
- Topological string/ spectral theory (TS/ST) correspondence
- ...

With the help of dualities, we can get interesting information on one side from simpler computation on the other side.

The TS/ST correspondence is in the frame of the topological string.
[Codesido, Grassi, Mariño, Hatsuda]

The topological string theories are the simplified version of the ordinary physical string theories with the topological twist (A and B twists).
[Witten]

We focus on non-compact toric Calabi-Yau threefolds X. They admit a natural physical description.

Topological string captures the information of the Gromov-Witten invariants.

Start with a toric CY X in the A model. We calculate the topological string free energy

$$F^{\mathrm{top}} = \sum_{g=0}^{\infty} g_s^{2-2g} F_g^{\mathrm{top}},$$

where g_s is the string coupling constant.
E.g.

$$F_0^{\mathrm{top}} = \sum_{\boldsymbol{d}} N_0^{\boldsymbol{d}} \mathrm{e}^{-\boldsymbol{d}\cdot\boldsymbol{t}},$$

where $\boldsymbol{t}$ denotes the Kähler moduli and $N_0^{\boldsymbol{d}}$ are the Gromov-Witten invariants at genus 0 and degree $\boldsymbol{d} \in H_2(X, \mathbb{Z})$.

Correspondingly, the topological string partition function is

$$Z^{\mathrm{top}} = \exp(F^{\mathrm{top}}).$$

F^{top} is an asymptotic formal power series, where

$$F_g^{\mathrm{top}} \sim (2g-2)!.$$

[Drukker, Gross, Mariño, Periwal, Putrov, Shenker, $\cdots$]

The divergence implies that there is some non-perturbative effect missing.

The TS/ST correspondence provides a non-perturbative completion of F^{top} (or Z^{top}).

By mirror symmetry, this is equivalent to the B model on the mirror CY $\tilde{X}$ of the form

$$W(X, Y, \boldsymbol{\kappa}) = uv, \quad X, Y \in \mathbb{C}^*, \quad u, v \in \mathbb{C},$$

where $W(X, Y, \boldsymbol{\kappa})$ is a polynomial in X, Y, X^{-1}, Y^{-1} and the coefficients are the complex moduli $\boldsymbol{\kappa}$.

The complex moduli are mapped to Kähler moduli on the A side by the mirror map.

$\tilde{X}$ can be viewed as a fibration over the (X, Y) plane.

We focus on the mirror curve

$$\Sigma \equiv \{(X, Y) | W(X, Y, \boldsymbol{\kappa}) = 0\}$$

characterizing the degeneration locus.

In the TS/ST correspondence, we turn on a quantization parameter

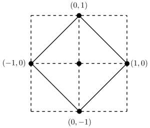
$$\hbar = \frac{1}{g_s}.$$

We are interested in the quantum mirror curve arising in the Weyl quantization of the mirror curve Σ.

We study their spectral properties, which are captured by the spectral determinant Ξ_X.

Example:

- A side: Local $\mathbb{P}_1 \times \mathbb{P}_1$.
- B side:



The mirror curve is

$$e^{y} + e^{-y} + \kappa_1 + \xi e^{-x} + e^{x} = 0, \quad x, y \in \mathbb{C}.$$

The corresponding quantum mirror curve is

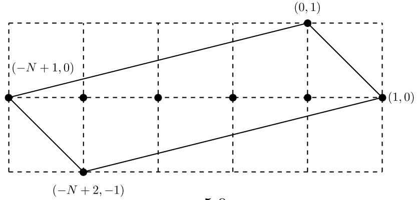
$$(e^{\hat{p}} + e^{-\hat{p}} + \kappa_1 + \xi e^{-\hat{x}} + e^{\hat{x}})\psi(x) = 0, \quad [\hat{x}, \hat{p}] = i\hbar.$$

Treating κ_1 as the energy, its spectrum is described by the spectral determinant:

$$\Xi_{\text{Local } \mathbb{P}_1 \times \mathbb{P}_1} = \det\left(1 + \kappa_1 (e^{\hat{p}} + e^{-\hat{p}} + \xi e^{-\hat{x}} + e^{\hat{x}})^{-1}\right)$$
$$= \prod_{n \geq 0} \left(1 + \frac{\kappa_1}{E_n}\right),$$

where E_n's are eigenvalues of $e^{\hat{p}} + e^{-\hat{p}} + \xi e^{-\hat{x}} + e^{\hat{x}}$.

In our work, we focus on $Y^{N,0}$ geometry, for $N \geq 2$.



$Y^{5,0}$

The corresponding quantum mirror curve is

$$\left(\mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}+(-N+2)\hat{x}} + \sum_{i=1}^{N-1} \kappa_{N-i} \mathrm{e}^{(i-N+1)\hat{x}} + \xi \mathrm{e}^{(-N+1)\hat{x}} + \mathrm{e}^{\hat{x}} \right) \psi(x) = 0,$$

where $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$. κ_i's are the complex moduli and ξ is a complex coupling.

We can construct $N-1$ non-commuting traceclass operators

$$A_i = \left(\mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}+(-N+2)\hat{x}} + \xi \mathrm{e}^{(-N+1)\hat{x}} + \mathrm{e}^{\hat{x}}\right)^{-1} \mathrm{e}^{-(i-1)\hat{x}},$$

one for each κ_i.

In the spectral theory, we define the spectral determinant:

$$\Xi_X(\boldsymbol{\kappa}, \hbar) = \det(1 + \kappa_1 A_1 + \cdots + \kappa_{N-1} A_{N-1}).$$

The TS/ST correspondence is a conjectured duality between the topological string and the spectral theory

$$\underbrace{\sum_{\boldsymbol{w}\in Q^{N-1}} \exp\left(\mathrm{J}_N(\boldsymbol{t}(g_s)+2\pi\mathrm{i}\boldsymbol{w},\xi,g_s)\right)}_{\text{topological string (A side)}} = \underbrace{\det\left(1+\sum_{i=1}^{N-1}\kappa_i A_i\right)}_{\text{spectral theory (B side)}},$$

where Q^{N-1} is the root lattice of A_{N-1}.
[Codesido, Hatsuda, Grassi, Mariño, $\cdots$]

- A side: J_N is determined by the topological string free energy and its non-perturbative completion.
- B side: The spectral determinant encodes the spectral information of the quantum mirror curve.

$$
\begin{aligned}
&\mathsf{J}_N(\boldsymbol{t}(g_s),\xi,g_s)\\
=&F^{\rm top}\left(g_s\xi+\pi \mathrm{i} N, g_s\boldsymbol{t}(g_s), g_s\right)\\
&+\sum_{i=1}^{N}\frac{t_i(g_s)}{2\pi}\frac{\partial}{\partial t_i}F^{\rm NS}_{\rm 5d}(\xi,\boldsymbol{t}(g_s),\frac{1}{g_s})+\frac{\partial}{\partial g_s}\left(g_s F^{\rm NS}_{\rm 5d}(\xi,\boldsymbol{t}(g_s),\frac{1}{g_s})\right).
\end{aligned}
$$

J_N contains two parts

- The topological string free energy: $F^{\rm top}$.
- The Nekrasov-Shatashvili free energy: $F^{\rm NS}_{\rm 5d}$.

[Grassi, Hatsuda, Mariño, Moriyama, Okuyama,$\cdots$]

$F^{\rm NS}_{\rm 5d}(\hbar)$ is a special function defined in 5d, $\mathcal{N}=1$ supersymmetric field theories.

It provides the non-perturbative effect in the TS/ST correspondence.

On the A side, topological string geometrically engineers 5d, $\mathcal{N} = 1$ supersymmetric field theory.
[Jefferson, Katz, Kim, Vafa, Klemm, $\cdots$]

Topological string partition functions are equivalent to the Nekrasov partition functions $Z_{5\mathrm{d}}^{\mathrm{Nek}}(\epsilon_1, \epsilon_2)$ at the Gromov-Witten/Gopakumar-Vafa phase, $\epsilon_1 = -\epsilon_2 = g_s$.

Inspired by the Nekrasov partition function, people study the topological string theory with two parameters ϵ_1 and ϵ_2, known as refined topological string theory.
[Huang, Kashani-Poor, Klemm, Kozcaz, Iqbal, Vafa, $\cdots$]

- The Gromov-Witten/Gopakumar-Vafa phase (standard topological string): $Z^{\mathrm{top}}(g_s) = \mathrm{e}^{F^{\mathrm{top}}(g_s)} = Z_{5\mathrm{d}}^{\mathrm{Nek}}(g_s, -g_s)$.
- The Nekrasov-Shatashvili phase: $Z_{5\mathrm{d}}^{\mathrm{NS}}(\hbar) = Z_{5\mathrm{d}}^{\mathrm{Nek}}(\hbar, 0)$.

$$\underbrace{\sum_{\boldsymbol{w}\in Q^{N-1}} \exp\left(\mathrm{J}_N(\boldsymbol{t}(g_s)+2\pi\mathrm{i}\boldsymbol{w},\xi,g_s)\right)}_{\text{topological string (A side)}} = \underbrace{\det\left(1+\sum_{i=1}^{N-1}\kappa_i A_i\right)}_{\text{spectral theory (B side)}}$$

The parameters on the two sides can be mapped as follows:

- $\hbar = \frac{1}{g_s}$.

  This gives a strong-weak duality.

- $\boldsymbol{\kappa}$ is mapped to $\boldsymbol{t}(g_s)$, where the map depends on g_s (or $\hbar$).

  This is the quantum version of the mirror map.

Although remained as a conjecture, the TS/ST correspondence has been tested in many examples and applied in a lot of works.

- Exact quantization conditions for the relativistic integrable systems.
  [Grassi, Gu, Klemm, Hatsuda, Huang, Mariño, Franco, Sun, Wang, Zhang, $\cdots$]
- Rigorous result for $\hbar = 2\pi$.
  [Codesido, Grassi, Mariño, Kashaev, Sergeev, Kerr, Doran, Sinha Babu, $\cdots$]
- Studies on the resurgence properties corresponding to CY geometries also exploit the TS/ST correspondence.
  [Gu, Mariño, Rella, Schiappa, $\cdots$]
- $\ldots$

A full proof of the TS/ST correspondence is still missing. In our work, we take a step forward to this final goal.

In today's talk, we prove a scaling limit of it.

Outline

A proof in the dual 4d limit

The large time expansion

Future directions

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On the A side, J_N contains two parts, Z^{top} and $Z^{\mathrm{NS}}_{\mathrm{5d}}$.

We are interested in the dual 4d limit where $Z^{\mathrm{NS}}_{\mathrm{5d}}$ vanishes.

We scale

$$\begin{aligned} g_s &= \beta \\ \log \xi &= -\frac{1}{\beta} \log(\beta^{2N} T) \\ \boldsymbol{t} &= 2\pi \mathrm{i} \boldsymbol{\sigma} \end{aligned}$$

and send $\beta \to 0$.

$$\sum_{\boldsymbol{w} \in Q^{N-1}} \exp\left(\mathrm{J}_N(\boldsymbol{t}(g_s) + 2\pi\mathrm{i}\boldsymbol{w}, \xi, g_s)\right) \xrightarrow[\text{4d limit}]{\text{dual}} \sum_{\boldsymbol{w} \in Q^{N-1}} Z^{\mathrm{SD}}_{\mathrm{4d}}(\boldsymbol{\sigma} + \boldsymbol{w}, T)$$

$Z^{\mathrm{SD}}_{\mathrm{4d}}(\boldsymbol{\sigma}, T)$ remained on the A side is another special function in physics.

It is the self-dual phase of the Nekrasov partition function for the 4d, $\mathcal{N} = 2$ $SU(N)$ SYM.

$$Z^{\mathrm{SD}}_{\mathrm{4d}}(\boldsymbol{\sigma}, T) = \frac{T^{\frac{1}{2}\boldsymbol{\sigma}^2} Z^{\mathrm{4d}}_{\mathrm{inst}}(\boldsymbol{\sigma}, T)}{\prod_{\boldsymbol{\alpha}\in\boldsymbol{\Delta}} G\left(1 + (\boldsymbol{\alpha}, \boldsymbol{\sigma})\right)},$$

where G is the Barnes function.

$Z^{\mathrm{4d}}_{\mathrm{inst}}(\boldsymbol{\sigma}, T)$ is the 4d Nekrasov instanton partition function. Schematically,

$$Z^{\mathrm{4d}}_{\mathrm{inst}}(\boldsymbol{\sigma}, T) = 1 + \sum_{n\geq 1} c_n(\boldsymbol{\sigma})\, T^n.$$

For $N = 2$,

$$
\begin{aligned}
& Z^{\mathrm{4d}}_{\mathrm{inst}}(\sigma_1, T) \\
=& 1 + \frac{1}{2\sigma_1^2} T \\
& + \frac{8\sigma_1^2 + 1}{4\sigma_1^2 \left(1 - 4\sigma_1^2\right)^2} T^2 \\
& + \frac{8\sigma_1^4 - 5\sigma_1^2 + 3}{24 \left(4\sigma_1^5 - 5\sigma_1^3 + \sigma_1\right)^2} T^3 \\
& + \cdots .
\end{aligned}
$$

$Z^{\mathrm{4d}}_{\mathrm{inst}}(\sigma_1, T)$ is a series with non-zero radius of convergence.
[Its, Lisovyy, Tykhyy]

On the B side, we scale

$$\hbar = \frac{1}{\beta}, \quad \log \xi = -\frac{\log(\beta^{2N} T)}{\beta},$$
$$\log \kappa_j = -\frac{j}{\beta N} \log\left(\beta^{2N} T\right) + \log\left(x_j\right).$$

Sending $\beta \to 0$,

$$\det\left(1 + \sum_{i=1}^{N-1} \kappa_i A_i\right) \xrightarrow[\text{4d limit}]{\text{dual}} \det\left(1 + \sum_{k=1}^{N-1} x_k A_k^{\rm 4d}\right),$$

where $A_k^{\rm 4d}$'s are traceclass operators defined by

$$A_k^{\rm 4d} = \mathrm{e}^{\frac{2k-N}{2N}\hat{p}} f(\hat{x}) \frac{1}{2\cosh(\frac{\hat{p}}{2})} f(\hat{x}), \quad k = 1, \cdots, N-1,$$

where

$$f(x) = \exp\left(-2NT^{\frac{1}{2N}} \cosh(x)\right)$$

[Bonelli, Grassi, Tanzini]

We take the dual 4d limit of the TS/ST correspondence for $Y^{N,0}$ geometry.

$$\sum_{\boldsymbol{w}\in Q^{N-1}} \exp\left(\mathrm{J}_N(\boldsymbol{t}(g_s)+2\pi\mathrm{i}\boldsymbol{w},\xi,g_s)\right) = \det\left(1+\sum_{i=1}^{N-1}\kappa_i A_i\right)$$

$$g_s = \beta, \quad \log\xi = -\frac{\log(\beta^{2N}T)}{\beta}$$

$$\boldsymbol{t} = 2\pi\mathrm{i}\boldsymbol{\sigma}$$

$$\hbar = \frac{1}{\beta}, \quad \log\xi = -\frac{\log(\beta^{2N}T)}{\beta},$$

$$\log\kappa_j = -\frac{j}{\beta N}\log\left(\beta^{2N}T\right) + \log\left(x_j\right).$$

$\beta \to 0$

$$\sum_{\boldsymbol{w}\in Q^{N-1}} Z^{\mathrm{SD}}_{4\mathrm{d}}(\boldsymbol{\sigma}+\boldsymbol{w},T) = \det\left(1+\sum_{k=1}^{N-1}x_k A^{4\mathrm{d}}_k\right).$$

[Gavrylenko, Grassi, H]

$$\sum_{\boldsymbol{w}\in Q^{N-1}} Z_{\mathrm{4d}}^{\mathrm{top}}(\boldsymbol{\sigma}+\boldsymbol{w},T) = \det\left(1+\sum_{k=1}^{N-1} x_k A_k^{\mathrm{4d}}\right)$$

A side depends on σ_i's while B side depends on x_k's, they are related by the k-th elementary symmetric function

$$x_k = \sum_{1\le i_1<i_2<\cdots<i_k\le N} \prod_{m=1}^{k} \mathrm{e}^{2\pi\mathrm{i}\sigma_{i_m}}, \quad k=1,\cdots,N-1.$$

Our strategy:

We match the A side constructed from the Nekrasov partition function with a solution, τ_0, of the rank N (non-autonomous) Toda equations. T is time in this language.
[Bershtein, Gavrylenko, Marshakov]

$$\partial^2_{\log T} \log \tau_j = -T^{\frac{1}{N}} \frac{\tau_{j+1}\tau_{j-1}}{\tau_j^2}, \quad \tau_j = \tau_{N+j}.$$

This is a generalization of the Kyiv formula. For $N = 2$, the Toda system reduces to the Painléve III_3.
[Gamayun, Iorgov, Lisovyy, $\cdots$]

In our proof, we show that the B side is the same solution to the Toda system as the A side by converting it to a known determinant solution and check its initial conditions.
[Tracy, Widom]

To generalize the result to the full TS/ST correspondence, we use the (Non-autonomous) q-Toda system as a bridge connecting two sides.

$$\mathcal{T}_j(qz)\mathcal{T}_j(\frac{z}{q}) = \mathcal{T}_j(z)^2 - z^{\frac{1}{N}}\mathcal{T}_{j+1}(z)\mathcal{T}_{j-1}(z)\ , \quad q = \mathrm{e}^{\mathrm{i}g_s}, \quad z \sim \xi^{-\frac{1}{\hbar}}.$$

In the limit dual 4d limit, it becomes a Toda system.

Using [Bershtein, Gavrylenko, Marshakov], we can prove the A side satisfies this equation. But proving it for the B side is challenging. [Gavrylenko, Grassi, H]

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(Non-autonomous) Toda equations are solved using 4d Nekrasov partition functions.

It is a series in time (small T) with a non-zero radius of convergence.

The spectral determinant on the B side provides a resummation of it.

A solution is specified by $2(N-1)$ initial conditions.

The 2 sides of the dual 4d limit corresponds to a particular type of initial conditions, where half of the initial conditions are fixed.

The remaining $(N-1)$ is parameterized by $\boldsymbol{\sigma}$ on the A side and $\boldsymbol{x}$ on the B side.

The spectral determinants have the small energies expansion

$$\det\left(1+\sum_{k=1}^{N-1} x_k A_k^{\mathrm{4d}}\right) = \sum_{M_1,\dots,M_{N-1}\geq 0} Z(\boldsymbol{M}) x_1^{M_1}\cdots x_{N-1}^{M_{N-1}}.$$

$Z(\boldsymbol{M})$ can be further recasted into a matrix model

$$Z(\boldsymbol{M}) = \frac{1}{M_1!\cdots M_{N-1}!}\int_{\mathbb{R}^M}\frac{d^M x}{(2\pi)^M}\prod_{j=1}^{N-1}\prod_{r_{j-1}\leq i_j\leq r_j} \mathrm{e}^{-T^{\frac{1}{2N}}\sin(\frac{\pi j}{N})\cosh(x_{i_j})}$$
$$\times\frac{\prod_{1\leq i<j\leq M} 2\sinh\left(\frac{x_i-x_j}{2}+\frac{1}{2}(d_i-d_j)\right) 2\sinh\left(\frac{x_i-x_j}{2}+\frac{1}{2}(f_i-f_j)\right)}{\prod_{i,j=1}^{M} 2\cosh\left(\frac{x_i-x_j}{2}+\frac{1}{2}(d_i-f_j)\right)}.$$

[Bonelli-Grassi-Tanzini]

The matrix model $Z(\boldsymbol{M})$ admits a natural expansion around the Gaussian point.

$$Z(\boldsymbol{M}) \sim \mathrm{e}^{-T^{\frac{1}{2N}}(\mathrm{M},\sin(\frac{\pi \mathrm{k}}{N}))} \left(T^{\frac{1}{2N}}\right)^{-\frac{1}{2}\boldsymbol{M}^2} C(\mathrm{M})\, \mathcal{E}^{\infty}(\mathrm{M}),$$

where

$$\mathcal{E}^{\infty}(\mathrm{M}) = 1 + \sum_{\ell \geq 1} \left(\frac{1}{T^{\frac{1}{2N}}}\right)^{\ell} D_{\ell}^{(N)}(\boldsymbol{M})\,.$$

C is a product of Barnes functions and $D_{\ell}^{(N)}$'s are polynomials of degree at most $3l$ in M.

This provides an expansion at large time (large T) for the solution to the Toda system.

The large time expansion can be further analytically continued to generic initial conditions, parametrized by $\boldsymbol{x}$ and $\boldsymbol{\nu}$:

$$\begin{aligned}
&\tau_0^{\infty}(\boldsymbol{x}, \boldsymbol{\nu}, T) \\
&= \sum_{\mathrm{M}\in\mathbb{Z}^{N-1}} \Big(\boldsymbol{x}^{\mathrm{M}+\boldsymbol{\nu}} \mathrm{e}^{-T^{\frac{1}{2N}}(\mathrm{M}+\boldsymbol{\nu}, \sin\frac{\pi \mathrm{k}}{N})} \big(T^{\frac{1}{2N}}\big)^{-\frac{1}{2}(\mathrm{M}+\boldsymbol{\nu})^2} \\
&\quad \times C(\mathrm{M}+\boldsymbol{\nu}) \sum_{\ell=0}^{\infty} \frac{D_\ell^{(N)}(\mathrm{M}+\boldsymbol{\nu})}{(T^{\frac{1}{2N}})^{\ell}}\Big) ,
\end{aligned}$$

[Gavrylenko, Grassi, H]

$\sum_{\ell=0}^{\infty} \frac{D_\ell^{(N)}(\mathsf{M}+\boldsymbol{\nu})}{(T^{\frac{1}{2N}})^\ell}$ provides the partition function of 4d $\mathcal{N}=2$ $SU(N)$ SYMs at strong coupling (large time).

The dual 4d limit of the TS/ST correspondence enables us to go from the weak coupling region, where the localization technique of Nekrasov is valid, to the strong coupling region.

It agrees with previous results at strong coupling.
[Bonelli, D'Hoker, Grassi, Klemm, Lerche, Phong, Tanzini, Theisen, $\cdots$]

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- Provide a proof to the TS/ST correspondence.
- Find a combinatorial expression for the coefficients $D_I^{(N)}$ characterizing the large time expansion of the solution.
- Get the generic mapping between small and large time initial conditions. Parallel to what has been done for $Y^{2,0}$, we expect a geometrical interpretation via Fock-Goncharov and Fenchel-Nielsen coordinates transform.
  [Coman, Longhi, Teschner]
- Explain the relation to q-Toda equations from the 5d, $\mathcal{N} = 1$ theory side. We expect it to be the tt^* equations of a codim-2 defect.
  [Bonelli, Cecotti, Gaiotto, Globlek, Moore, Neitzke, Tanzini, Vafa]

Thank you!

The kernel of A_i can be expressed explicitly in terms of the Faddeev's quantum dilogarithm Φ_b:

$$A_j(p,p') = \mathrm{e}^{-\mathrm{i}\pi b^2 (j-1)^2/N^2} \mathrm{e}^{-4\pi(j-1)bp'/N} \rho_{1,N-2,\xi}(p, p' + \mathrm{i}\frac{b(j-1)}{N}),$$

where $b^2 = \frac{N\hbar}{2\pi}$ and

$$\rho_{1,N-2,\xi}(p,p') = \frac{\overline{f_{\mathrm{5d}}(p)} f_{\mathrm{5d}}(p')}{2b\cosh\left(\pi\frac{p-p'}{b} + \frac{\mathrm{i}\pi(N-2)}{2N}\right)}$$

where

$$f_{5d}(x) = \frac{\Phi_b(x - \frac{1}{2\pi b}\log\xi + \frac{\mathrm{i}b}{2N})}{\Phi_b(x - \frac{\mathrm{i}b(N-1)}{2N})} \mathrm{e}^{\frac{\pi b(N-1)}{N}x} \mathrm{e}^{-\frac{1}{2N}\log\xi}.$$

$Z(\boldsymbol{M})$'s are the fermionic spectral traces given by

$$Z(\boldsymbol{M}) = \frac{1}{M_1! \cdots M_{N-1}!} \sum_{\sigma \in S_M} (-1)^\sigma \int \mathrm{d}^M x \left(\prod_{i=1}^{M_1} A_1^{\mathrm{4d}}(x_{\sigma(i)}, x_i) \right)$$
$$\left(\prod_{i=1+M_1}^{M_1+M_2} A_2^{\mathrm{4d}}(x_{\sigma(i)}, x_i) \right) \cdots \left(\prod_{i=1+\cdots+M_{N-2}}^{M_1+\cdots+M_{N-1}} A_{N-1}^{\mathrm{4d}}(x_{\sigma(i)}, x_i) \right)$$

$$r_0 = 1, \quad r_j = \sum_{i=1}^{j} M_i \quad j = 1, 2, \cdots .$$

We also define

$$d_j = -\frac{(N-1-k)\mathrm{i}\pi}{N},$$
$$f_j = -\frac{(N-2)\mathrm{i}\pi}{N} - d_j,$$

where

$$r_{k-1} \leq j \leq r_k.$$

$$D_1^{(3)} = -\frac{2M_1^3}{3\sqrt{3}} + \frac{1}{2}\sqrt{3}M_2 M_1^2 + \frac{1}{2}\sqrt{3}M_2^2 M_1 + \frac{5M_1}{12\sqrt{3}} - \frac{2M_2^3}{3\sqrt{3}} + \frac{5M_2}{12\sqrt{3}}.$$

$$\begin{aligned} D_2^{(3)} = {} & \frac{2M_1^6}{27} - \frac{1}{3}M_2 M_1^5 + \frac{1}{24}M_2^2 M_1^4 + \frac{17M_1^4}{54} + \frac{97}{108}M_2^3 M_1^3 - \frac{389}{216}M_2 M_1^3 + \frac{1}{24}M_2^4 M_1^2 \\ & - \frac{4}{3}M_2^2 M_1^2 - \frac{85M_1^2}{288} - \frac{1}{3}M_2^5 M_1 - \frac{389}{216}M_2^3 M_1 + \frac{493M_2 M_1}{432} + \frac{2M_2^6}{27} + \frac{17M_2^4}{54} - \frac{85M_2^2}{288}. \end{aligned}$$

$$\begin{aligned} D_3^{(3)} = {} & -\frac{4M_1^9}{243\sqrt{3}} - \frac{13M_1^7}{54\sqrt{3}} - \frac{439M_1^5}{432\sqrt{3}} + \frac{32021M_1^3}{31104\sqrt{3}} + \frac{7M_1}{144\sqrt{3}} + \frac{7M_2}{144\sqrt{3}} - \frac{18689M_2^2 M_1}{3456\sqrt{3}} \\ & + \frac{M_2 M_1^8}{9\sqrt{3}} + \frac{577M_2 M_1^6}{324\sqrt{3}} + \frac{43133M_2 M_1^4}{5184\sqrt{3}} + \frac{13429M_2^2 M_1^3}{1728\sqrt{3}} - \frac{18689M_2 M_1^2}{3456\sqrt{3}} \\ & - \frac{5M_2^2 M_1^7}{36\sqrt{3}} - \frac{13}{32}\sqrt{3}M_2^2 M_1^5 - \frac{10633M_2^3 M_1^4}{2592\sqrt{3}} + \frac{13429M_2^3 M_1^2}{1728\sqrt{3}} + \frac{32021M_2^3}{31104\sqrt{3}} \\ & - \frac{469M_2^3 M_1^6}{1296\sqrt{3}} + \frac{77M_2^4 M_1^5}{144\sqrt{3}} - \frac{10633M_2^4 M_1^3}{2592\sqrt{3}} - \frac{13}{32}\sqrt{3}M_2^5 M_1^2 + \frac{43133M_2^4 M_1}{5184\sqrt{3}} - \frac{439M_2^5}{432\sqrt{3}} \\ & - \frac{4M_2^9}{243\sqrt{3}} + \frac{M_1 M_2^8}{9\sqrt{3}} - \frac{5M_1^2 M_2^7}{36\sqrt{3}} - \frac{13M_2^7}{54\sqrt{3}} - \frac{469M_1^3 M_2^6}{1296\sqrt{3}} + \frac{577M_1 M_2^6}{324\sqrt{3}} + \frac{77M_1^4 M_2^5}{144\sqrt{3}}. \end{aligned}$$