THE ASYMPTOTICS AND THE DESCENDANTS OF THE 3D-INDEX

RENEW QUANTUM SEMINAR

ØDENSE, 7 FEB 2023

STAVROS GAROUFALIDIS
MATH INSTITUTE
SUSTECH, SHENZHEN, CHINA

References:

2209.02843 Periods, the mero 3D-index and TV-invariants

2301.00098 The descendants of the 3D-index

Contents:

- The 3d-index and the state-integral
- The 3d-index (definitions, properties)
- Descendants (algebra)
- Asymptotics (analysis)

The 3d-index and the state integral

3d-3d correspondence on one side gives an $N=2$ superconformal field theory

$T_M[S^3]$ on $S^3 = D^2 \times S^1 \cup_S D^2 \times S^1$ $\quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2\mathbb{Z}$

or on

$T_M[S^2 \times S^1]$ on $S^2 \times S^1 = D^2 \times S^1 \cup D^2 \times S^1$ $\quad \mathcal{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL_2\mathbb{Z}$

and more generally on

$T_M[\text{Lens space}]$ $\quad \gamma \in SL_2\mathbb{Z}$

- Here, $M =$ triangulated 3-manifold.

- These theories are topological, and their partition functions can be computed by cut-and-paste topology, reducing the computation to the partition function of an (ideal) tetrahedron.

- The partition function of an ideal tetrahedron satisfies
  - a pair of linear q-difference equations
  - a nonlinear pentagon identity

that uniquely determines them, and the whole theory
symmetries

- $T_M[S^3]$ = Andersen-Kashaev state integral, expressed (conjecturally) bilinearly in terms of functions of q and $\tilde{q}$ where $q = e^{2\pi i \tau}$, $\tilde{q} = e^{-2\pi i/\tau}$

- $T_M[S^2 \times S^1]$ = DGG-index, expressed in terms of functions of q and q^{-1}

- T_M[Lens space] is a state-integral defined using an $SL_2\mathbb{Z}$-extension of the Faddeev quantum dilogarithm [GKZ] expressed bilinearly in terms of the same functions of q and $q^{\#}$
$$q = e^{2\pi i \tau}, \quad q^{\#} = e^{2\pi i \gamma(\tau)}.$$

The DGG 3d-index

Building block: tetrahedron index

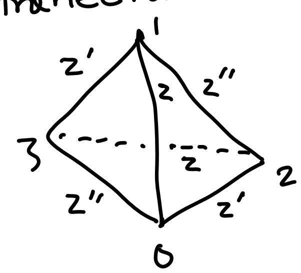
$$I_\Delta(m,e)(q) = \sum_{n=(-e)_+}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)-(n+\frac{1}{2}e)m}}{(q;q)_n (q;q)_{n+e}} \in \text{"}\mathbb{Z}((q))\text{"}$$

$(m)_+ = \max\{0,m\}$. The partition function of $I^{rot}_\Delta[S^2\times S^1]$

$$\deg(I_\Delta(m,e)(q)) = \frac{1}{2}\Big(m_+(m+e)_+ + (-m)_+e_+ + (-e)_+(-e-m)_+ + \max\{0,m,-e\}\Big) \geq 0$$

Rotated 3d-index

$\mathcal{T}$ = ideal triangulation of M (cusped 3-mfld) with N tetrahedra



$$z' = \frac{1}{1-z}$$

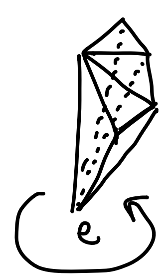
$$z'' = 1 - \frac{1}{z}$$

Note $\begin{cases} z z' z'' = -1 \\ z^{-1} + z'' = 1 \end{cases}$

Gluing equations: N for edges, 2 for cusp

$$\sum_{j=1}^{N} G_{ij}\log z_j + G'_{ij} z'_j + G''_{ij}\log z''_j = \pi i \eta_i \qquad i=1,\dots,N+2$$

$$\eta = \begin{pmatrix} 2 \\ \vdots \\ 2 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{Z}^{N+2}$$



Eliminate z' using $z' = -\frac{1}{zz''}$

$$\sum_{j=1}^{N} A_{ij} \log z + B_{ij} \log z_j'' = \pi i \nu_i \qquad i = 1 \ldots, N+2$$

Let $\quad A = (a_1 | \ldots | a_N) \qquad B = (b_1 | \ldots | b_N)$

So $\quad \mathcal{T} \rightsquigarrow (A | B | \nu)$

Def $\quad I_{\mathcal{T}}^{rot}[S^1 \times S^2]$ computed via 3d-3d correspondence, a topological theory

Def $\quad I_{\mathcal{T}}^{rot}(n, n')(q) = \sum_{k \in \mathbb{Z}^N} S_{\mathcal{T}}(k, n, n')(q)$

$$S_{\mathcal{T}}(k, n, n')(q) = (-q^{1/2})^{\nu \cdot k - (n-n')\nu_\lambda} q^{k_N(n+n')/2} \cdot \prod_{j=1}^{N} I_{\Delta}(a_j''(n-n') - b_j \cdot k, -a_j(n-n') + a_j \cdot k)(q)$$

Thm [GHRS] (a) If $\mathcal{T}$ = 1-efficient (ie has no nonperipheral normal tori, and no normal S^2) then

$I_{\mathcal{T}}^{rot}(n, n')(q) \in \mathbb{Z}((q))$ is well-defined

(b) It is a topological invariant of cusped hyperbolic manifolds

Thm [GK] Alternative proof of topological invariance using state-integral formula $I_{\mathcal{T}}^{mer}$

In particular, $I^{mer}_{\tau}(0,0)(q) = \sum_{n\in\mathbb{Z}} I^{rot}_{\tau}(n,n)(q)$

Properties of 3d-index (Conjectural)

Factorization

$I^{rot}_{\tau}[S^1\times S^2] = \langle I^{rot}_{\tau}[S^1\times D^2], I^{rot}_{\tau}[S^1\times D^2]\rangle$

- $\underset{\mathbb{Z}\times\mathbb{Z}}{I^{rot}_{\tau}(q)} = \underset{\mathbb{Z}\times r}{H_{\tau}(q)}\,\underset{r\times r}{B_{\tau}}\,\underset{r\times\mathbb{Z}}{H_{\tau}(q^{-1})^t}$

$H_{\tau}(q) = (h^{(\alpha)}_n(q)) \qquad \alpha=1..r, \; n\in\mathbb{Z}$

$\downarrow$ colored holo blocks

- where $H(q)$ is a properly normalized fundamental solution of linear q-difference equation

$\hat{A}_{\tau}(M_+,L_+)H(q)=0 \qquad \hat{A}_{\tau}(M_-,L_-)H(q^{-1})=0$

$M_+ h^{(\alpha)}_n(q) = q^n h^{(\alpha)}_n(q) \qquad M_- h^{(\alpha)}_n(q^{-1}) = q^{-n} h^{(\alpha)}_n(q^{-1})$

$L_+ h^{(\alpha)}_n(q) = h^{(\alpha)}_{n+1}(q) \qquad L_- h^{(\alpha)}_n(q^{-1}) = h^{(\alpha)}_{n+1}(q^{-1})$

Cor $I^{rot}_{\tau}(q)$ is uniquely determined by

(1) $r\times r$ matrix $I^{rot}_{\tau}(q)[r]$

(2) pair of linear q-difference eqns

$$I^{rot}(q)[r] = \left(I^{rot}(n,n')(q) \right)_{0 \le n,n' \le r-1}$$

Regularity

$$I^{rot}_{\tau}(n,n')(q) = \lim_{x \to 1} \sum_{\alpha} B^{(\alpha)}_{\tau}(q^{-n'}x^{-1}; q^{-1})\, B^{(\alpha)}_{\tau}(q^{n}x; q)$$

where $B^{(\alpha)}(x;q)$ are x-deformed holo blocks

Descendants

Insertions, defects, line operators

$M = S^3 - K$ $\longleftrightarrow$ $\mathcal{O} \in \mathcal{M}(T)$

$L \subset M$

$$\mathcal{M}(\tau) = \frac{W_q(\tau)}{(W_q(\tau)\cdot \text{Lagrangians} + \text{edge eqns}\cdot W_q(\tau))}$$

$W_q(\tau) = \mathbb{Q}(q) \langle \hat{z}_j, \hat{z}_j', \hat{z}_j'' \mid i = 1 \dots N \rangle$ = q-Weyl algebra

$\hat{z}\hat{z}' = q\hat{z}'\hat{z}$ + cyclic permutations

$\hat{z}\hat{z}'\hat{z}'' = -q$

Def If $\mathcal{O} = \prod_{j=1}^{N} \hat{z}_j^{\alpha_j} \hat{z}_j''^{b_j}$

$$I^{rot}_{\tau,\mathcal{O}}(n,n')(q) = \sum_{k \in \mathbb{Z}^N} (\mathcal{O} \circ S_\tau)(k,n,n')(q)$$

$$(\mathcal{O} \circ S_\tau)(k,n,n')(q) = (-q^{1/2})^{\nu \cdot k - (n-n')\nu_\lambda} q^{k\nu(n+n')/2 + L_{\mathcal{O}}(n,n';k)}$$

$$\cdot \prod_{j=1}^{N} I_\Delta(\lambda_j''(n-n') - b_j \cdot k + b_j, -\lambda_j(n-n') + a_j \cdot k - \alpha_j)(q)$$

Note The insertion $\mathcal{E}_i$ of the ith edge acts as

$$(\mathcal{E}_i \circ S_\tau)(k,n,n') = q S_\tau(k - e_i, n, n')$$

hence summing over $k \in \mathbb{Z}^N$, it follows $\mathcal{E}_i - q$ annihilates $I^{rot}_\tau(n,n')(q)$.

Properties of descendant 3d-index (Conjectural)

Factorization ⓐ There exists $\hat{A}_{\tau,\theta}$ linear q-difference operator with fund matrix solution $H_{\tau,\theta}(q)$:

$$I^{rot}_{\tau,\theta}(q) = H_{\tau,\theta}(q) B_\tau H_\tau(q^{-1})^t$$

ⓑ There exists $Q_{\tau,\theta}(q) \in M_r(\mathbb{Q}(q^{1/2}))$:

$$I^{rot}_{\tau,\theta}[r] = Q_{\tau,\theta}(q) I^{rot}_\tau[r]$$

Cor $I^{rot}_{\tau,\theta}(q)$ is uniquely determined by

(a) $r \times r$ matrices $I^{rot}_{\tau}(q)[r]$ (vacuum) and $Q_{\tau,\theta}(q)$

and

(b) pair of linear q-difference eqns $\hat{A}_{\tau}$ and $\hat{A}_{\tau,\theta}$.

Def let $DI^{rot}_{\tau,\theta} = \mathrm{Span}_{\mathbb{Q}(q^{1/2})} \{ I^{rot}_{\tau,\theta}(n,n')(q) \mid n,n' \in \mathbb{Z} \}$

(a fin. dimensional $\mathbb{Q}(q^{1/2})$-vct space)

Cor $\bigcup_{\theta \in \mathcal{M}(\tau)} DI^{rot}_{\tau,\theta} = DI^{rot}_{\tau}$.

In other words, the descendants of the rotated 3d index are expressed effectively by a finite size matrix over $\mathbb{Q}(q^{1/2})$.

Conjecture $\hat{A}_{\tau,\theta}(M,L,q)\big|_{q=1} \doteq_M \hat{A}_{\tau}(M,L)$

Examples

4_1 knot

$$G = \begin{pmatrix} 2 & 2 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \quad G' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} \quad G'' = \begin{pmatrix} 0 & 0 \\ 2 & 2 \\ 0 & -1 \\ 1 & -3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$I^{rot}_{4_1}(n,n')(q) = \sum_{k_1,k_2 \in \mathbb{Z}} q^{\frac{k_2(n+n')}{2}} I_\Delta(k_1, k_1+k_2)(q) I_\Delta(k_1+k_2-n+n', k_1-n+n')(q)$$

Factorization

$$I^{rot}(n,n')(q) = -\frac{1}{2} h^{(1)}_{n'}(q^{-1}) h^{(0)}(q) + \frac{1}{2} h^{(0)}_n(q^{-1}) h^{(1)}_{n'}(q)$$

$$h^{(0)}_n(q) = (-1)^n q^{|n|(2|n|+1)/2} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2+|n|k}}{(q;q)_k (q;q)_{k+2|n|}}$$

$h^{(1)}_n(q)$ = similar, a bit more complicated

satisfy linear q-difference eqn ($n \in \mathbb{Z}$, $\alpha = 0,1$):

$$q^{2+2n}(q^{3+2n}-1) h^{(\alpha)}_n(q) + \dots + q^{3+2n}(q^{1+2n}-1) h^{(\alpha)}_{n+2}(q) = 0$$

Defects $\mathcal{O} = -\hat{z}_1^{-1} - \hat{z}_2^{-1} + \hat{z}_1^{-1}\hat{z}_2^{-1} \in \mathcal{U}(\mathcal{T})$

Computing the matrices $I^{\mathrm{rot}}(n,n')(q)$ and $I^{\mathrm{rot}}_{\hat{\mathcal{O}}}(n,n')(q)$ for $n,n'=0,1$ we find out

$$I^{\mathrm{rot}}(n,n')(q)[2] = \begin{pmatrix} 1-8q-9q^2+18q^3+46q^4+90q^5+\dots & -q^{-\frac{1}{2}}+q^{\frac{1}{2}}-q^{\frac{3}{2}}+6q^{\frac{5}{2}}+20q^{\frac{7}{2}}+\dots \\ -q^{-\frac{1}{2}}+q^{\frac{1}{2}}-q^{\frac{3}{2}}+6q^{\frac{5}{2}}+20q^{\frac{7}{2}}+\dots & 2q+2q^2+7q^3+8q^4+3q^5-22q^6+\dots \end{pmatrix}$$

and likewise

$$I^{\mathrm{rot}}_{\hat{\mathcal{O}}}(n,n')(q)[2] = \begin{pmatrix} -3+15q+24q^2-15q^3-69q^4+\dots & 2q^{-\frac{1}{2}}-q^{\frac{1}{2}}+4q^{\frac{3}{2}}-7q^{\frac{5}{2}}-34q^{\frac{7}{2}}+\dots \\ q^{-\frac{3}{2}}-q^{-\frac{1}{2}}-q^{\frac{1}{2}}+q^{\frac{3}{2}}-5q^{\frac{5}{2}}+\dots & -1-2q-4q^2-9q^3-17q^4-13q^5+\dots \end{pmatrix}$$

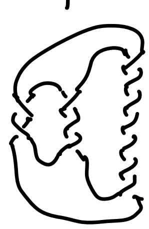
satisfy

$$I^{\mathrm{rot}}_{\hat{\mathcal{O}}}(n,n')(q)[2] = \frac{1}{1-q}\begin{pmatrix} q-2 & q^{\frac{1}{2}} \\ -q^{\frac{1}{2}} & q+1-q^{-1} \end{pmatrix} I^{\mathrm{rot}}(n,n')(q)[2] + O(q^{121})$$

illustrating the dramatic cancellation of the sum of products of q-series into a short rat function.

In particular

$$I^{\mathrm{rot}}_{\hat{\mathcal{O}}}(0,0)(q) = \frac{1}{1-q}\left((q-2)\,I^{\mathrm{rot}}(0,0)(q) + q^{\frac{1}{2}}\,I^{\mathrm{rot}}(0,1)(q)\right). \qquad \square$$

This was repeated for 5_2 knot ($r=3$, 3 tetrahedra) and even more impressively for the $(-2,3,7)$ pretzel knot  ($r=6$, 3 tetrahedra)

where $I^{rot}(q)[6]$ has 20 digits integer coefficients for the coeff of q^{160},
and so did $I^{rot}_{\hat{\Sigma}_2}(q)[6]$, however their ratio is a short rational function with coefficients integers between $-5, 5$.

Asymptotics

The story begins in 2011 on a train to Bonn with Don Zagier after a conference in the Diablerets

$$g(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n^2} = 1 - q - 2q^2 - 2q^3 - 2q^4 + q^6 + \dots$$

$$(q;q)_n = (1-q)(1-q^2)\dots(1-q^n)$$

Incidentally $g(q) = I_\Delta(0,0)(q)$ but I_Δ was introduced a few years later.

Radial asymptotics

$q = e^{2\pi i \tau} = e(\tau)$, $e(x) = e^{2\pi i x}$

- $\tau \downarrow 0$ vertically

$$g(e^{-\frac{2\pi}{N}}) \sim \frac{1}{\sqrt{N}} \frac{\sqrt{2}}{\sqrt[4]{3}} \left(\cos \frac{\mathrm{vol}}{2\pi} N + \sin \frac{\mathrm{vol}}{2\pi} N \right) \left(1 - \frac{11}{72\sqrt{3}} \frac{2\pi}{N} + \frac{697}{2(72\sqrt{3})^2} \left(\frac{2\pi}{N}\right)^2 + \dots \right)$$

First the period $\frac{\mathrm{vol}}{2\pi} \sim 0.32$ of the oscillation was guessed, then 697 was found by search in files regarding the volume conjecture of the 4_1 knot.

$$\mathrm{vol} = \mathrm{vol}(4_1) = 2\, \mathrm{Im}\, \mathrm{Li}_2(e(\tfrac{1}{6}))$$

Thus

$$g(e(\tau)) \sim \zeta_8 \sqrt{\tau} \left(\hat{\Phi}(2\pi i \tau) - i \hat{\Phi}(-2\pi i \tau) \right)$$

where $\Phi(h) = e^{\frac{i\,\mathrm{vol}(4_1)}{h}} O(h)$

$$\Phi(h) = \frac{1}{\sqrt[4]{-3}} \left(1 - \frac{11}{72\sqrt{-3}} h + \frac{697}{2(72\sqrt{-3})^2} h^2 + \frac{724351}{30(72\sqrt{-3})^3} h^3 + \dots \right)$$

satisfies

$$\langle 4_1 \rangle_N \sim N^{3/2} \hat{\Phi}\left(\frac{2\pi i}{N} \right)$$

$\langle 4_1 \rangle_N =$ Kashaev invariant of 4_1

- $\tau \to 0$ horizontally $0 < \arg \tau < \varepsilon$ fixed

  Here $\hat{\Phi}(2\pi i \tau)$ is exponentially bigger than $\hat{\Phi}(-2\pi i \tau)$

  but now $\tilde{q}$ corrections are visible, $\tilde{q} = e^{-2\pi i/\tau}$

  and in fact

$$g(e(\tau)) \sim \zeta_8 \sqrt{\tau}\, g(e(-\tfrac{1}{\tau}))\, \hat{\Phi}(2\pi i \tau)$$

Actually this needs to be interpreted eg via median Borel summation. In this case $\hat{\Phi}(-2\pi i \tau)\, h(e(-1/\tau))$ also contributes and then have identity.

<u>Conclusion</u> Radial asymptotics of q-hypergeometric series at $q \to$ root of unity depends on sectors.

Fast forward to last year.

$$I^{mer}_{\mathcal{T}}(0,0)(q) = \sum_{n\in\mathbb{Z}} I^{rot}_{\mathcal{T}}(n,n)(q)$$

(the relation between the [GK] 3d-index and the rotated 3d-index)

$$I^{mer}_{4_1}(0,0)(q) = 1-4q-q^2+36q^3+70q^4+100q^5+34q^6-116q^7+\cdots$$

when $q=e(\tau)$, $\tau\to 0$ <u>nearly horizontally</u> we found that

$$I^{mer}_{4_1}(0,0)(q) \sim e^{\frac{2\,\mathrm{vol}}{2\pi\tau}} \frac{1}{3^{3/4}\,2^{1/2}\sqrt{\tau}}$$

$$\times\left(1-\frac{19}{24\sqrt{-3}^{3}}2\pi i\tau+\frac{1333}{1152\sqrt{-3}^{6}}(2\pi i\tau)^2-\frac{1601717}{414720\sqrt{-3}^{9}}(2\pi i\tau)^3+\cdots\right)$$

Then, we recognized the number 1333 appearing in the asymptotics of the TV invariant $TV_{4_1,m+\frac{1}{2}}$ replacing $m+\frac12$ by τ. Then we checked 7 more coefficients.

<u>Def</u> $$TV_{K,m+\frac12} = * \sum_{k=1}^{m} [k]^2 |J_{K,k}(q)|^2 \Big|_{q=e\left(\frac{1}{m+\frac12}\right)}$$

Chen-Yang conjectured that TV_K grows exponentially capturing the volume of a hyperbolic knot.

<u>Conjecture</u> [GW] $\tau \to 0$ nearly horizontally

$$I_K^{mer}(0,0)(e(\tau)) \sim \sum_{n\in\mathbb{Z}} \hat{\Phi}^{(\sigma_1)}_{K,n}(2\pi i\tau)\, \hat{\Phi}^{(\sigma_2)}_{K,n}(-2\pi i\tau)$$

$$TV_{K,m+\frac{1}{2}} \sim \sum_{n\in\mathbb{Z}} \hat{\Phi}^{(\sigma_1)}_{K,n}\left(\frac{2\pi i}{m+\frac{1}{2}}\right) \hat{\Phi}^{(\sigma_2)}_{K,n}\left(-\frac{2\pi i}{m+\frac{1}{2}}\right)$$

$$I_K^{rot}(n,n')(e(\tau)) \sim \hat{\Phi}^{(\sigma_1)}_{K,n}(2\pi i\tau)\, \hat{\Phi}^{(\sigma_2)}_{K,n'}(-2\pi i\tau)$$

where $\hat{\Phi}^{(\sigma)}_n(\hbar) = e^{\frac{V_\sigma}{\hbar}} \Phi^{(\sigma)}_n(\hbar)$

$$\Phi^{(\sigma)}_n(\hbar) = \frac{1}{\sqrt{\delta_\sigma}}\left(1 + F_\sigma[[n,\hbar]]\right)$$

$\sigma = \sigma_1$ = geometric representation

$\sigma = \sigma_2$ = complex conjugate

and $\left(\hat{\Phi}^{(\sigma)}_n(\hbar)\right)$ is a fundamental solution to the $\hat{A}$-eqn annihilating $I^{rot}(q)$.

$\hat{\Phi}^{(\sigma)}_n(\hbar)$ can be computed (once $\hat{A}$ is known) by WKB.

This conjectured was confirmed for 4_1 and 5_2 in [GW].

Vertical asymptotics

Conjecture [GW] When $\tau \downarrow 0$ vertically

$$I^{rot}(n,n')(e(\tau)) \sim \sum_\sigma \varepsilon_\sigma \hat{\Phi}^{(\sigma)}_n(2\pi i\tau)\, \hat{\Phi}^{(\bar{\sigma})}_{n'}(-2\pi i\tau)$$

$$\text{where } \varepsilon_\sigma = \begin{cases} -1 & \text{if } \delta_\sigma < 0 \\ 1 & \text{otherwise} \end{cases}$$

Note that after cancelling the exponential factors, the constant terms of the power series are positive numbers. The asymptotic series are $O(\tau^{poly})$ perhaps a consequence of unitarity.

Now vertical asymptotics $I^{mer}_{4_1}(0,0)(q)$

Numerically, we found out that

$$I^{mer}_{4_1}(0,0)(q) \sim e^{\frac{2\,vol}{2\pi\tau}} \frac{1}{3^{3/4}\, 2^{1/2} \sqrt{\tau}}$$

$$\times \left(1 - \frac{19}{24\sqrt{-3}^{3}} 2\pi i\tau + \frac{1333}{1152\sqrt{-3}^{6}}(2\pi i\tau)^2 - \frac{1601717}{414720\sqrt{-3}^{9}}(2\pi i\tau)^3 + \dots\right)$$

$$-i \times (\text{replace } \tau \text{ by } -\tau \text{ in } \uparrow)$$

$$+ \kappa_{4_1} \frac{i}{\tau} + \kappa'_{4_1}\, i\tau + \dots$$

$$\text{where } \kappa_{4_1} = 0.4458257949935614977$$

From previous experience we tried to recognize this as an alg number and π but failed. Then Campbell suggested to look for periods of A-poly curve

$$y = x^{-2} - 2x^{-1} - 1 - 2x + x^2 \quad \text{(elliptic curve)}$$

and much to our surprise

$$\varkappa_{4_1} = \frac{\varpi_{4_1} + \overline{\varpi}_{4_1}}{2\pi}$$

$$\varpi_{4_1} = \int_{\mathbb{R}} \frac{du}{\sqrt{e^{-2u} - 2e^{-u} - 1 - 2e^{u} + e^{2u}}}.$$

(period of differential of second kind).

Likewise $\varkappa'_{4_1} = 0.1005975490738001278\,9$

was also recognized as

$$\varkappa'_{4_1} = 2\pi(\varpi'_{4_1} + \overline{\varpi}'_{4_1})$$

$$\varpi'_{4_1} = \int_C \frac{e^{-3u} - e^{-2u} - 2e^{-u} + 5 - 2e^{-u} - e^{-2u} + e^{3u}}{(e^{-2u} - 2e^{-u} - 1 - 2e^{u} + e^{2u})^{7/2}}\, du$$

Moreover, next 10 coeffs are $\mathbb{Q}$-lin. combs of ϖ_{4_1} and ϖ'_{4_1}.

This success was repeated for the 5_2 knot whose A-poly is the curve

$$y^2 = (1-y)(1-xy)\left(1-\frac{y}{x}\right) \quad \text{(also elliptic)}.$$

Periods are Mellin-Barnes integrals

Thm [GK]

$$\frac{1}{2\pi i}\int_{\varepsilon - i\mathbb{R}} \frac{B(z,z)^2}{(\cos\pi z)^2}\,dz = 2\int_{-\infty}^{1}\frac{dx}{\sqrt{(1-x)(1-x+4x^2)}} = 5.60241216$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^x(1-t)^y\,\frac{dt}{t(1-t)}$$

is the Euler B-function.