Difference equations and 3–manifolds

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Campbell Wheeler

December 6th, 2022

Contents

Based on...

Joint work with Stavros Garoufalidis, Jie Gu, and Marcos Mariño.

Motivation

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Recent, work of Garoufalidis, Gu and Mariño described the picture for knots in some simple examples. Their picture generalises to certain kinds of hypergeometric series.

I aim to explain what happens when one deals with closed three manifolds and how this relates to the $\hat{Z}$ invariants of Gukov et. al. that are predicted from physics.

The full package

One hopes for matrix valued invariants

- which solve difference equations
- whose ratios at $\hbar$ and $4\pi^2/\hbar$ give Borel resummations of asymptotic series
- whose ratios at $\hbar$ and $-\hbar$ give Stokes constants of asymptotic series

q–difference equations

What are q–difference equations?

Consider, the q–Weyl algebra over $\mathbb{Z}[q^{\pm}]$ generated by the symbols x, σ such that

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We think of q as formal or for most of this talk $|q| < 1$. This algebra acts on functions in x and q by

$$(xf)(x;q) = xf(x;q) \quad \text{and} \quad (\sigma f)(x;q) = f(qx;q) .$$

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Then a linear q–difference equation of order r is an equation for f of the form

$$\left(\sum_{k=0}^{r} a_k(x;q)\sigma^k\right) f \;=\; 0\,.$$

This can be thought of as defining a left module of the q–Weyl algebra.

Solving q–difference equations: Newton polygon

To solve linear q-difference equations formally we can use an analogue of Frobenius' method for differential equations. To apply this algorithm it is most instructive to consider the Newton polygon. This stores many of the important structures associated to difference equations.

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Definition:

The Newton polygon of an operator

$$\sum_{k,\ell} a_{k,\ell} x^{\ell} \sigma^{k}$$

is the convex hull of the set of points $(k,\ell) \in \mathbb{R}^2$ such that $a_{k,\ell} \neq 0$.

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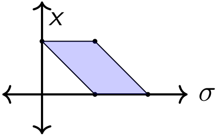
For example, the operator $(\sigma - 1)(\sigma + x) = \sigma^2 + (qx - 1)\sigma - x$



Solving q–difference equations: Convention

If we have a function f that satisfies a q-difference equation and a function g such that

$$(\sigma - 1)g \;=\; 0 \quad \text{that is} \quad g(qx;q) = g(x;q)$$

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For our considerations a natural choice is given by

$$\theta(x;q) \;=\; \sum_{k\in\mathbb{Z}}(-1)^k q^{k(k+1)/2}x^k \;=\; (qx;q)_\infty(x^{-1};q)_\infty(q;q)_\infty$$

where

$$(x;q)_n \;=\; \prod_{j=0}^{n-1}(1-q^jx)\,.$$

Solving q–difference equations: Ansatz

Now Frobenius' method for linear differential equations takes an ansatz of the form

$$\exp(p(x^{-1}))\sum_{k=0}\alpha_k x^{k+r}\,,$$

where p is some polynomial. To solve a linear q-difference equation we take the following ansatz

$$\theta(x^{\kappa};q^{\kappa})\sum_{k=0}^{\infty}\alpha_k(q)x^k\frac{\theta(\rho^{-1}x;q)}{\theta(x;q)}\,.$$

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We choose κ so that it correspond to a slope of the Newton polygon. Then the vanishing of α_k for $k < 0$ will give a polynomial equation for ρ called the indicial polynomial. Once these are solved we get a recursion for α_k determined by α_0. If ρ has multiplicity then we take an expansion $\rho = \rho_0 e^{\epsilon}$.

Solving q–difference equations: Wronskians and Divergence

If we find a basis of solutions to a r order difference equation $f_1(x;q),\dots,f_r(x;q)$ then we can put them into a matrix called the Wronskian which satisfies a first order matrix q-difference equation.

$$W(x;q) = \begin{pmatrix} f_1(x;q) & f_2(x;q) & \dots & f_r(x;q) \\ f_1(qx;q) & f_2(qx;q) & \dots & f_r(qx;q) \\ \vdots & \vdots & \dots & \vdots \\ f_1(q^{r-1}x;q) & f_2(q^{r-1}x;q) & \dots & f_r(q^{r-1}x;q) \end{pmatrix}$$

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The slopes of the Newton polygon also determine the behaviour of the coefficients α_k. If we have negative slopes with largest absolute value κ (and we are solving for a flat edge) then

$$\alpha_k(q) \sim q^{-\kappa k^2/2+\beta k} O(1)\,.$$

Therefore, if we want to solve in meromorphic functions we would like a way to convert these divergent solutions to convergent. This can be done by q–Borel re-summation as proved by Dreyfus.

Solving q–difference equations: Borel resummation

The q-Borel transform is defined

$$\mathcal{B}_\kappa \sum_{k=0}^{\infty} \alpha_k(q) x^k = \sum_{k=0}^{\infty} (-1)^k q^{\kappa k(k+1)/2} \alpha_k(q) \xi^k \,,$$

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and the q–Laplace transform is defined

$$(\mathcal{L}_\kappa f)(x, \lambda; q) = \sum_{\ell \in \mathbb{Z}} \frac{f(q^{\kappa\ell} \lambda^\kappa x; q)}{\theta(q^{\kappa\ell} \lambda^\kappa; q^\kappa)} \,.$$

Note it is elliptic in λ.

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If $f(x; q)$ is a polynomial in x then

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If $f(x; q)$ is a polynomial in x then

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Proof:

$$\sum_{\ell \in \mathbb{Z}} \frac{(q^{\kappa\ell} \lambda^\kappa x)^k}{\theta(q^{\kappa\ell} \lambda^\kappa; q^\kappa)} = \frac{x^k}{\theta(\lambda^\kappa; q^\kappa)} \sum_{\ell \in \mathbb{Z}} (-1)^\ell q^{\kappa\ell(\ell+1)/2 + \kappa\ell k} \lambda^{\kappa(\ell+k)} = (-1)^k q^{-\kappa k(k+1)/2} x^k \,.$$

Aside: q–Stokes phenomenon and monodromy

The fact that the q–Laplace transform depends on an additional elliptic variables is often referred to as q–Stokes phenomenon. It is often a little mysterious to describe. However, in the kind of examples we will consider later, we expect that this should be computable.

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$$M(x;q) = W_0(x,\mu;q)^{-1} W_\infty(x,\lambda;q)\,.$$

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Theorem:[Zwegers]

The operator $(\sigma - 1)(\sigma + x)$ has monodromy

$$M = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

where

$$* = -\frac{(q;q)_\infty^3 \theta(q^{-1}t;q)\theta(\lambda^{-1}\mu;q)\theta(\lambda^{-1}\mu^{-1}t^{-1};q)}{\theta(\lambda^{-1};q)\theta(\mu;q)\theta(\lambda^{-1}t^{-1};q)\theta(\mu^{-1}t^{-1};q)}$$

3–manifolds

TQFTs: Witten's path integral

In the 80s, Jones discovered a remarkable invariant of links. Witten interpreted the Jones polynomial in terms of quantum field theory. In particular, for $\mathrm{SU}(2)$ connections on a three manifold $\mathcal{A}_M$,

$$Z_M(\hbar) = \int_{\mathcal{A}_M/\mathcal{G}_M} \exp\left(\frac{CS(A)}{2\pi i\hbar}\right) DA$$

where

$$CS(A) = \int_M \mathrm{Tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \in \mathbb{C} + (2\pi i)^2 \mathbb{Z}\,.$$

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Therefore, $1/\hbar \in \mathbb{Z}$ for the integral to be well defined. Witten then studies Wilson loops observables for embedded knots and conjectures the existence of the coloured Jones polynomial for knots K and $N \in \mathbb{Z}_{>0}$ where $q = \exp(2\pi i\hbar)$

$$J_N(K;q) \in \mathbb{Z}[q^{\pm}]\,.$$

The critical points of $CS(A)$ are flat connections,

$$\mathcal{A}_M^{\mathrm{flat}}/\mathcal{G}_M \cong \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})\,.$$

Witten–Reshetikhin–Turaev invariants: R–matrix and coloured Jones

Soon after Reshetikhin–Turaev realised what should be Witten's theory mathematically. Firstly, they construct the coloured Jones polynomials from universal R–matrices of quantum groups $U_q\mathfrak{sl}_2$. They use this to construct representations of the braid group. To each component they label by a representation and to each crossing they associated an R matrix from the representation.

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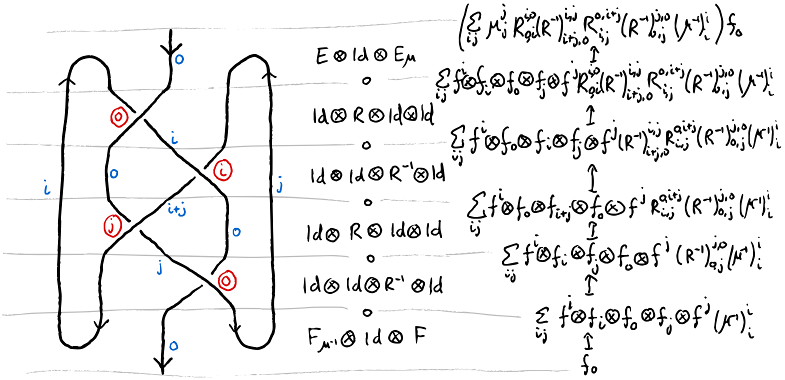
$V_m = \mathrm{Span}\{e_{-m}, e_{-m+1}, \cdots, e_{m-1}, e_m\}$,
$V_n = \mathrm{Span}\{e_{-n}, e_{-n+1}, \cdots, e_{n-1}, e_n\}$, $\mu : V_n \to V_n$ such that
$\mu(e_j) = \sum_i \mu_j^i e_i = q^j e_j$, $R : V_n \otimes V_m \to V_m \otimes V_n$ such that

$$R(e_k \otimes e_\ell) = \sum_{i=-m}^{m} \sum_{j=-n}^{n} R_{k\ell}^{ij} e_i \otimes e_j = \sum_{i=-m}^{m} \sum_{j=-n}^{n} \sum_{p=0}^{\min(m-i, j+n)} \delta_{\ell, i+p} \delta_{k+p, j}$$

$$\times (-1)^p q^{ij - \frac{p}{2}(m+n) - (i-j)p - p(p+1)/2} \frac{(q;q)_{m+\ell}(q;q)_{n-k}}{(q;q)_{m+i}(q;q)_p(q;q)_{n-j}} e_i \otimes e_j \,.$$

Example: 4_1



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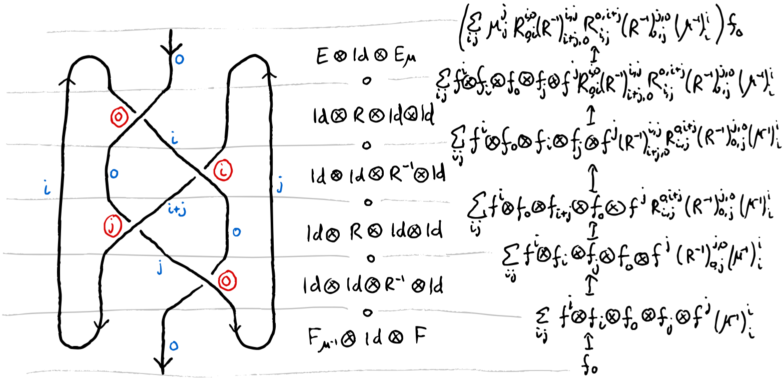

$$J_N(4_1;q) \;=\; \sum_{k=0}^{N-1} q^{-kN}(q^{N-1};q^{-1})_k(q^{N+1};q)_k\,.$$

To difference operators: $\hat{A}$

Using Zeilberger theory Garoufalidis and Lê proved the existence of a recursion for the coloured Jones polynomial. This follows from the explicit form of the R–matrix.

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$$(\sigma J)_N(q) \;=\; J_{N+1}(q) \quad \text{and} \quad (xJ)_N(q) = q^N J_N(q)\,.$$

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Even before the proof of its existence Garoufalidis and Gukov conjectured that taking the classical limit of $\hat{A}$ defines a variety of flat $\mathrm{SL}_2(\mathbb{C})$ connection connections on $S^3 - K$. This is called the AJ conjecture.

Example: 4_1

The coloured Jones polynomial for 4_1 satisfies the following recursion

$$\begin{aligned} J_{N+1}(4_1;q) \;=\; & q^{-N}\frac{(1+q^N)(1-q^{2N+1})}{1-q^{N+1}} \\ & -q^{-2N}\frac{(q^N-1)^2(q^N+1)(q^{4N+1}-q^{3N+1}-q^{2N+2}-q^{2N}-q^{N+1}+q)}{(1-q^{N+1})(q^{2N}-q)}J_N(4_1;q) \\ & -\frac{1-q^{2N+1}}{1-q^{2N-1}}\frac{1-q^{N-1}}{1-q^{N+1}}J_{N-1}(4_1;q)\,. \end{aligned}$$

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This equation can be homogenised to a third order equation. Taking $q^N = m^2$ and replacing $J_{N+a}(K;q)$ by ℓ^a in the homogeneous part of the second order equation we get the polynomial equation

$$\ell - (m^{-4} - m^{-2} - 2 - m^2 + m^4) + \ell^{-1} \;=\; 0$$

Witten–Reshetikhin–Turaev invariants: Closed manifolds

To construct invariants of closed 3–manifolds Witten suggested a surgery formula from links.

Theorem:[Lickorish–Wallace]

Every closed three manifold can be constructed from gluing solid tori into link complements.

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Theorem:[Lickorish–Wallace]

Every closed three manifold can be constructed from gluing solid tori into link complements.

Reshetikhin–Turaev gave the formula using the coloured Jones and used Kirby calculus to prove its invariance. For a framed ℓ component link L and a root of unity $q^{1/4} = \exp(2\pi i a/4c)$ they take

$$\begin{aligned} WRT(M;q) \;=\; & \left(\frac{2\exp(-\pi i/4)(q^{1/2}-q^{-1/2})}{\sum_{k=1}^{4c} q^{k^2/4}}\right)^{\ell} \exp\left(\frac{-3\pi i\sigma_L}{4}\right) q^{3\sigma_L/4} \\ & \times \sum_{N_1,\cdots,N_\ell=1}^{c-1} J_{N_1,\cdots,N_\ell}(L;q) \prod_{j=1}^{\ell}\left(\frac{q^{N_j/2}-q^{-N_j/2}}{q^{1/2}-q^{-1/2}}\right)^2 . \end{aligned}$$

Example: $4_1(1,2)$

The manifold $4_1(-1,2)$ is a hyperbolic integer homology sphere. Beliakov and Lê prove the following formula

$$(1-q)WRT(q) \;=\; \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^k q^{-k(k+1)/2+\ell(\ell+1)}\frac{(q;q)_{2k+1}}{(q;q)_{\ell}(q;q)_{k-\ell}}\,.$$

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We can add an extra variable as this expression is a hypergeometric function to get

$$F_m(q) \;=\; \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^k q^{-k(k+1)/2+\ell(\ell+1)+mk} \frac{(q;q)_{2k+1}}{(q;q)_{\ell}(q;q)_{k-\ell}}\,.$$

Example: $4_1(1,2)$

This defines a non–canonical module associated to the difference equation

$$\begin{aligned}
q^{2m+2}F_m(q) + (q^{2m+4} + q^{m+1} + q^{m+2})F_{m+1}(q) \\
+(-q^{2m+7} - q^{2m+5} - q^{2m+4} + q^{m+3} + 1)F_{m+2}(q) \\
+(-q^{2m+9} - q^{2m+7} - q^{2m+6} + q^{m+n+3} - q^{m+5} - q^{m+4} - q^{m+3})F_{m+3}(q) \\
+(q^{2m+10} + q^{2m+9} + q^{2m+7} + q^{m+n+4} - q^{m+6})F_{m+4}(q) \\
+(q^{2m+12} + q^{2m+11} + q^{2m+9} - q^{m+n+6} + q^{m+6})F_{m+5}(q) \\
+(-q^{2m+12} - q^{m+n+7})F_{m+6}(q) - q^{2m+14}F_{m+7}(q) \;=\; (1-q)\,.
\end{aligned}$$

Example: $4_1(1,2)$

This defines a non–canonical module associated to the difference equation

$$\begin{aligned} q^{2m+2}F_m(q) + (q^{2m+4} + q^{m+1} + q^{m+2})F_{m+1}(q) \\ +(-q^{2m+7} - q^{2m+5} - q^{2m+4} + q^{m+3} + 1)F_{m+2}(q) \\ +(-q^{2m+9} - q^{2m+7} - q^{2m+6} + q^{m+n+3} - q^{m+5} - q^{m+4} - q^{m+3})F_{m+3}(q) \\ +(q^{2m+10} + q^{2m+9} + q^{2m+7} + q^{m+n+4} - q^{m+6})F_{m+4}(q) \\ +(q^{2m+12} + q^{2m+11} + q^{2m+9} - q^{m+n+6} + q^{m+6})F_{m+5}(q) \\ +(-q^{2m+12} - q^{m+n+7})F_{m+6}(q) - q^{2m+14}F_{m+7}(q) \;=\; (1-q)\,. \end{aligned}$$

The classical limit of this gives the equation

$$z^7 + 2z^6 - 3z^5 - 3z^4 + 5z^3 + z^2 - 3z - 1 \;=\; 0\,.$$

This variety (field) gives the set of flat $\mathrm{SL}_2(\mathbb{C})$ connections of this manifold.

Two variable series: solving $\hat{A}$ in q–series

Recently, Gukov and Manolescu proposed the existence of q series with a Jacobi like variable x associated to knots they denote

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They state that these series should be related to a resummation of asymptotic series of Vassilev invariants. Concretely, they give a method of calculation by taking the $\hat{A}$ operator and some initial conditions from the coloured Jones polynomial and solve the recursion in q-series.

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Notice that $(\ell - 1)$ is always a factor in the A polynomial and therefore from the AJ conjecture one should always be able to solve the recursions as a power series in x as there should be a horizontal edge of the Newton polygon.

Example: 4_1

Recently, Park gave a formula for the figure eight knot

$$F_{4_1}(x;q) = \frac{1}{2}\sum_{k,j,\ell=0}^{\infty}\left(x^{k+j+\ell+1/2} - x^{k+j+\ell+3/2} - x^{-k-j-\ell-1/2} + x^{-k-j-\ell-3/2}\right) \times \binom{k+j}{j}_q \binom{k+\ell}{\ell}_{q^{-1}}$$

where

$$\binom{k}{\ell}_q = \frac{(q;q)_k}{(q;q)_\ell (q;q)_{k-\ell}}\,.$$

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&\qquad \times \binom{k+j}{j}_q \binom{k+\ell}{\ell}_{q^{-1}}
\end{aligned}
$$

where

$$
\binom{k}{\ell}_q = \frac{(q;q)_k}{(q;q)_\ell (q;q)_{k-\ell}}\,.
$$

Earlier, around 2010, Garoufalidis and Zagier found a version of this series which can thought of a some kind of Kashaev invariant q–series. This is also related to the holomorphic blocks of Beem, Dimofte, Pasquetti.

$$
g(q) = \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q;q)_k^2}\,.
$$

$\hat{Z}$ and the WRT invariant

Gukov and collaborators expect q–series invariants of closed three manifolds (and a spin structure a) they denote as $\hat{Z}_a(q)$. These should be related to some kind of categorification of the WRT invariant of the closed three manifold. Although there is still no definition Gukov and Manolescu propose a surgery formula using the analogue of the Laplace transform of Beliakov and Lê to construct q–series from their two variable series.

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$$\mathcal{L}^{(a)}_{p/r} : x^u q^v \mapsto \begin{cases} q^{-u^2 r/p} q^v & \text{if } ru - a \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} .$$

For a knot K and a rational number p/r they take

$$\hat{Z}_a(K(p,r);q) \;=\; \pm q^{\mathbb{Q}} \mathcal{L}^{(a)}_{p/r}((x^{1/2r} - x^{-1/2r}) F_K(x;q)) .$$

Example: $4_1(1,2)$

Using the formula

$$(1-q)WRT(q) \;=\; \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^k q^{-k(k+1)/2+\ell(\ell+1)}\frac{(q;q)_{2k+1}}{(q;q)_\ell(q;q)_{k-\ell}}\,.$$

I considered an associated q-series

$$\begin{aligned}\hat{Z}(q) \;&=\; \sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell}(-1)^{k+\ell} q^{\frac{1}{2}3k(k+1)+\frac{1}{2}\ell(\ell+1)-k}\frac{(q;q)_\ell}{(q;q)_{2k}(q;q)_{\ell-k}}\\ &=\; 1-q+2q^3-2q^6+q^9+3q^{10}+q^{11}-q^{14}-3q^{15}+\dots\,.\end{aligned}$$

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Comparing this to the formula of $\hat{Z}$ I realised they agreed up to initial constants.

Proposition:

This q-series is the same as computed by Gukov and Manolescu.

Asymptotics

Stationary phase

We can apply stationary phase approximations to Witten's integral

$$Z_M(\hbar) \;=\; \int_{\mathcal{A}_M/\mathcal{G}_M} \exp\left(\frac{CS(A)}{2\pi i\hbar}\right) DA\,.$$

This is not defined but if Witten's ideas are correct and the manifold has isolated non-degenerate flat connections, we expect to find asymptotics of the form as $\hbar \to 0$

$$WRT(\hbar) \;\sim \sum_{A \in \text{flat } \mathrm{SU}(2) \text{ connections}} \exp\left(\frac{CS(A)}{2\pi i\hbar}\right)\Phi_A(2\pi i\hbar)\,.$$

This asymptotic expansion is known as Witten's asymptotic expansion conjecture.

Volumes conjecture for knots

In the early 90s, Kashaev constructed an invariant of links for each root of unity q and conjectured that it should have asymptotics as $q \to 1$ that grow exponentially like the volume. Murakhami–Murakhami showed that Kashaev's invariant was a specialisation of the coloured Jones polynomial which provided the following conjecture called the volume conjecture.

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Conjecture:

For $N \in \mathbb{Z}$ as $N \to \infty$

$$J_N(K; \exp(2\pi i/N)) \sim \exp(\mathrm{Vol}_{\mathbb{C}}(K)N/2\pi i)\Phi_K(2\pi i/N)\,.$$

Work of Dimofte, Garoufalidis, Gukov, Hikami, Lennels, Zagier provided a conjectural definition of Φ using triangulations.

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This should correspond to a geometric connection, which is an $\mathrm{SL}(\mathbb{C})$ connection. Physically, this arises when we think of Witten's integral as a contour integral inside the complexified space of $\mathrm{SU}(2)$ connections which can be related to $\mathrm{SL}_2(\mathbb{C})$ connections.

Volume conjecture for closed 3–manifolds

Chen–Yang proposed a volume conjecture for WRT invariants. In particular, they suggest that

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For $N \in \mathbb{Z}$ as $N \to \infty$

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This is quite different to Witten's asymptotic expansion conjecture. However, we will see in a moment they are part of one conjecture.

Example: 4_1

For 4_1 we have

$$\mathrm{Vol}_{\mathbb{C}} \;=\; 2.0299\cdots i$$

and numerically (and in this case proved by many) we find that for $N \in \mathbb{Z}$ and as $N \to \infty$

$$J_N(\exp(2\pi i/N)) \sim \exp(\mathrm{Vol}_{\mathbb{C}}(K)N/2\pi i)N^{3/2}\frac{1}{3^{1/4}}\Big(1+\frac{11}{24\sqrt{-3}^3}\frac{2\pi i}{N}+\cdots\Big)$$

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Similarly, Garoufalidis–Zagier found that for τ on a small angle above the reals

$$g(\exp(2\pi i/\tau)) \sim \exp(\mathrm{Vol}_{\mathbb{C}}(K)\tau/2\pi i)\tau^{-1/2}\frac{1}{3^{1/4}}\Big(1+\frac{11}{24\sqrt{-3}^{3}}\frac{2\pi i}{\tau}+\cdots\Big)$$

Example: $4_1(1,2)$

Now one can compute that this manifold has 7 flat connections which are one Galois orbit of the geometric connection defined over a degree 7 field and the trivial connection. The seven (ordered so the last two are complex embeddings corresponding to the geometric and antigeometric) complex volumes and one loops (constants of Φ) δ and generators of embbedings of the associated fields are

$$
\begin{aligned}
\mathrm{Vol}_{\rho_1} &= 20.297\ldots, & \delta_{\rho_1} &= -11.578\ldots, & \xi_{\rho_1} &= -2.2411\ldots \\
\mathrm{Vol}_{\rho_2} &= -6.7857\ldots, & \delta_{\rho_2} &= -12.636\ldots & \xi_{\rho_2} &= -0.43760\ldots \\
\mathrm{Vol}_{\rho_3} &= 39.362\ldots, & \delta_{\rho_3} &= -83.275\ldots, & \xi_{\rho_3} &= 0.25599\ldots \\
\mathrm{Vol}_{\rho_4} &= 9.2837\ldots, & \delta_{\rho_4} &= -7.0205\ldots, & \xi_{\rho_4} &= 1.3348\ldots \\
\mathrm{Vol}_{\rho_5} &= 2.1292\ldots, & \delta_{\rho_5} &= -5.3937\ldots, & \xi_{\rho_5} &= 1.3483\ldots \\
\mathrm{Vol}_{\rho_6} &= 4.8678\cdots - i1.3985\ldots, & \delta_{\rho_6} &= 3.9517\cdots - i0.15252\ldots, & \xi_{\rho_6} &= 0.36981\cdots - i1.4410\ldots \\
\mathrm{Vol}_{\rho_7} &= 4.8678\cdots + i1.3985\ldots, & \delta_{\rho_7} &= 3.9517\cdots + i0.15252\ldots, & \xi_{\rho_7} &= 0.36981\cdots + i1.4410\ldots\,.
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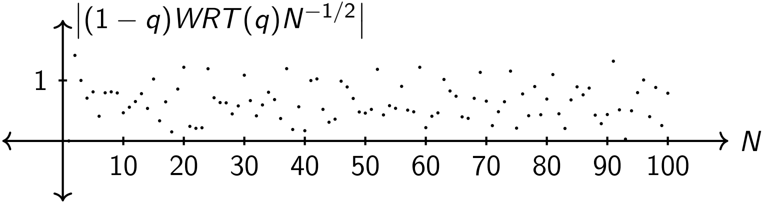
Witten's asymptotic expansion conjecture was studied by Andersen–Hansen and proved to leading order by Charles–Marché for this manifold. The $\mathrm{SU}(2)$ connections here correspond to $1,2,4,5$ and the trivial (which is polynomially smaller than the other four). We can plot the values against the leading order.



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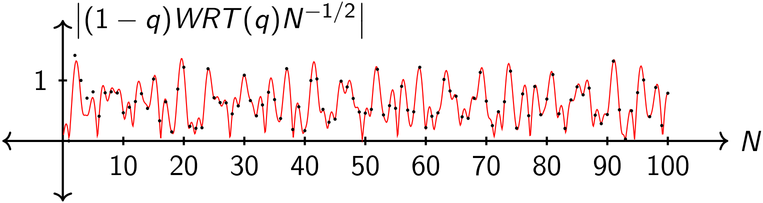

$$\Big| \exp\left(\frac{\mathrm{Vol}_1 N}{2\pi i}\right) \delta_1^{-1/2} \exp(3\pi i/4) + \exp\left(\frac{\mathrm{Vol}_2 N}{2\pi i}\right) \delta_2^{-1/2} \exp(-\pi i/4)$$

$$+ \exp\left(\frac{\mathrm{Vol}_4 N}{2\pi i}\right) \delta_4^{-1/2} \exp(-\pi i/4) + \exp\left(\frac{\mathrm{Vol}_5 N}{2\pi i}\right) \delta_5^{-1/2} \exp(3\pi i/4)\Big|$$

Example: $4_1(1,2)$

We find numerically (in this case it is proved by Ohtsuki) that for $N \in \mathbb{Z}$ and $N \to \infty$ and $q = \exp(2\pi i/(N+1/2))$

$$(1-q)WRT(q) \sim 2\exp(\mathrm{Vol}_7 N/2\pi i)N^{1/2}\frac{\exp(3/8)}{\delta_7^{1/2}}\Big(1+\cdots\Big)$$

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Similarly, for τ on a small angle above the reals we find

$$\hat{Z}(\exp(2\pi i/\tau)) \sim \exp(\mathrm{Vol}_7\tau/2\pi i)\tau^{1/2}\frac{\exp(3/8)}{\delta_7^{1/2}}\Big(1+\cdots\Big)$$

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while for $\tau \in i\mathbb{R}$ we see exponential growth

$$\hat{Z}(\exp(2\pi i/\tau)) \sim \exp(\mathrm{Vol}_3\tau/2\pi i)\tau^{1/2}\frac{\exp(3/8)}{\delta_3^{1/2}}\Big(1+\cdots\Big)$$

The dominant term will depend on the argument of τ.

A conterexample

Gukov and Manolescu conjecture that $\hat{Z}$ should have radial limits to the WRT invariant. There are a few issues with this conjecture. In particular, taking $q \to \exp(2\pi ia/c)$ on a very low angle should always see a dominant contribution from asymptotic series coming from the geometric connection for a hyperbolic manifold, which will dominate the asymptotics for any hyperbolic manifold. However, even taking strictly radial limits the previous example for $4_1(1,2)$ will have exponential growth as $q \to \exp(2\pi ia/c)$.

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This provides a counter example to the conjecture as stated however it is not entirely incorrect. These q-series will not necessarily have a limit but they seems to have asymptotic series determined by all flat connections. If the trivial connection dominates then the radial limit conjecture will be true however if another connection dominates then it will fail.

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This indicates that the conjecture should be replaced by an analogue of the volume conjecture in general.

Quantum modularity at roots of unity

Quantum modularity extends the asymptotic discussion to rational numbers as opposed to just integer values of N. We already saw a volume conjecture using rational values name the Chen–Yang volumes conjecture which evaluated at $N + 1/2$.

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$$J_{denom(1/N)}(4_1; \exp(2\pi i/N))$$

$$\sim J_{denom(-N)}(4_1; \exp(-2\pi i N)) \exp(\mathrm{Vol}_{\mathbb{C}}(K)N/2\pi i)N^{3/2}\frac{1}{3^{1/4}}\Big(1 + \frac{11}{24\sqrt{-3}^3}\frac{2\pi i}{N} + \cdots$$

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and for $q = \exp(2\pi i/N)$ and $\tilde{q} = \exp(-2\pi i N)$ and

$$(1-q) WRT(q) \sim (1-\tilde{q}) WRT(\tilde{q}) \exp(\mathrm{Vol}_7 N/2\pi i) N^{1/2} \frac{\exp(3/8)}{\delta_7^{1/2}} \Big(1 + \cdots\Big)$$

Of course when $N \in \mathbb{Z}$ and $\tilde{q} = 1$ then $(1-\tilde{q}) WRT(\tilde{q}) = 0$ and we get growth exponentially smaller than the volume. In fact, from Witten's conjecture we expect polynomial growth.

Quantum modularity for q–series

For the $q = \exp(2\pi i/\tau)$ series when the argument of τ is extremely small $\tilde{q} = \exp(-2\pi i\tau)$ is close to $O(1)$. Then we can observe (using say optimal truncation) that

$$g(q) \sim g(\tilde{q})\exp(\mathrm{Vol}_{\mathbb{C}}(K)\tau/2\pi i)\tau^{-1/2}\frac{1}{3^{1/4}}\left(1 + \frac{11}{24\sqrt{-3}^3}\frac{2\pi i}{\tau} + \cdots\right)$$

and

$$\hat{Z}(q) \sim \hat{Z}(\tilde{q})\exp(\mathrm{Vol}_7\tau/2\pi i)\tau^{1/2}\frac{\exp(3/8)}{\delta_7^{1/2}}\left(1 + \cdots\right).$$

Conjectures for Borel and Stokes

Borel resummation

To make analytic functions from factorially divergent series we can use Borel resummation.

$$f(x) \;=\; \sum_{k=0}^{\infty} a_k x^{\lambda+k}\,.$$

The Borel transform of this series is defined to be

$$\mathcal{B}_1 f(\xi) \;=\; \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(\lambda+k+1)}\xi^{\lambda+k}\,.$$

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If this function is convergent an has an analytic continuation with certain growth conditions at infinity, then we can take it's Laplace transform

$$\mathcal{L}_1\mathcal{B}_1 f(x) \;=\; \int_0^{\infty} \exp(-\xi)\mathcal{B}f(\xi x) d\xi\,.$$

This is called the Borel resummation of f which has the same asymptotics as $s(f)$ from Watson's lemma.

Refined quantum modularity

Using the Borel resummations we can find exponentially small corrections to the previous asymptotics expressions such as

$$(1-q)WRT(q) \; ? = ? \; \sum_{k=0}^{7} \omega_k(\tilde{q}) \exp(\mathrm{Vol}_k N/2\pi i) N^{1/2} \frac{\exp(3/8)}{\delta_k^{1/2}} s\Big(1 + \cdots\Big)$$

for some elements ω_k which behave similarly to the WRT invariant and correspond to special elements of the modules associate to the difference equations associated to $4_1(1,2)$.

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for some elements ω_k which behave similarly to the WRT invariant and correspond to special elements of the modules associate to the difference equations associated to $4_1(1,2)$. Using a full Wronskian matrix W_m we find that for a diagonal weight matrix Δ, P, Q matrices of rational functions

$$W_m(q) \;?=?\; s(\hat{\Phi}_m)(2\pi i/N) P(\tilde{q}) W_0(\tilde{q}) Q(\tilde{q}) \Delta(N)\,.$$

This allows to consider a conjectural formula for the Borel resummation

$$s(\hat{\Phi}_m)(2\pi i/N) \;?=?\; W_m(q)\Delta(N)^{-1} Q(\tilde{q})^{-1} W_0(\tilde{q})^{-1} P(\tilde{q})^{-1}$$

State integrals

The right had side of the equation can be proved to be given by integrals of a special function the Faddeev dilogarithm (for $e(z) = \exp(2\pi i z)$)

$$\mathcal{D}(z;\tau) \;=\; \frac{(e(z+\tau);e(\tau))_\infty}{(e(z/\tau);e(-1/\tau))_\infty}\,.$$

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Andersen and Kashaev constructed a TQFT for manifolds with cusps using integrals of this function. Regardless for the manifold $4_1(1,2)$ we can write down a state integral and this is of the form

$$-\frac{2(\tilde{q};\tilde{q})_\infty}{\tau^2(q;q)_\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{D}(z_2;\tau)\mathcal{D}(2z_1;\tau)\mathcal{D}(z_2-z_1;\tau)$$
$$\times\exp((z_1^2/\tau+z_1z_2/\tau+z_1(-m-m'/\tau)+z_2(1+1/\tau))2\pi i)dz_1dz_2\,.$$

This is an integral analogue of q–hypergeometric functions. It satisfies decoupled q and $\tilde{q}$ difference equations w.r.t. m and m'.

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$$-\frac{2(\tilde{q};\tilde{q})_\infty}{\tau^2(q;q)_\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{D}(z_2;\tau)\mathcal{D}(2z_1;\tau)\mathcal{D}(z_2-z_1;\tau)$$
$$\times\exp((z_1^2/\tau+z_1z_2/\tau+z_1(-m-m'/\tau)+z_2(1+1/\tau))2\pi i)dz_1dz_2\,.$$

This is an integral analogue of q–hypergeometric functions. It satisfies decoupled q and $\tilde{q}$ difference equations w.r.t. m and m'.

These identities provide analytic properties of the RHS. Finding appropriate P give conjectural formulae for the Borel resummation.

Conjectures for singularities of the Borel transform

There are conjectures for the singularities for Borel transform of asymptotic series. In particular, for the asymptotic series $\mathcal{B}_1\Phi_A$ associated to a connection A it is conjectured that the Borel transform with have logarithmic singularities at the points

$$CS(B) - CS(A) + 4\pi^2\mathbb{Z}.$$

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At each of these points the function looks locally like

$$S_{A,B,k}\exp((CS(B) - CS(A) + 4\pi^2 k)/(2\pi i z))$$
$$\times\mathcal{B}_1(\mathrm{Ei}((CS(B) - CS(A) + 4\pi^2 k)/z)\Phi_B(z))$$

where $S_{A,B,k} \in \mathbb{Z}$. These integers are called Stokes constants. The Borel resummation for $\arg(\tau) = \arg(CS(B) - CS(A) + 4\pi^2 k) \pm \epsilon$ changes by

$$S_{A,B,k}\exp((CS(B) - CS(A) + 4\pi^2 k)/(2\pi i z))s(\Phi_B)\,.$$

Conjectures for Stokes matrices

Therefore, we can collect Stokes constants into matrices of $\tilde{q}$ series indexed by

$$CS(A) + 4\pi^2\mathbb{Z}.$$

Then if we compute using quantum modularity for τ with some fixed argument just above the positive reals

$$s(\hat{\Phi}_m)(2\pi i/N) \;?=?\; W_m(q)\Delta(N)^{-1}Q_I(\tilde{q})^{-1}W_0(\tilde{q})^{-1}P_I(\tilde{q})^{-1}$$

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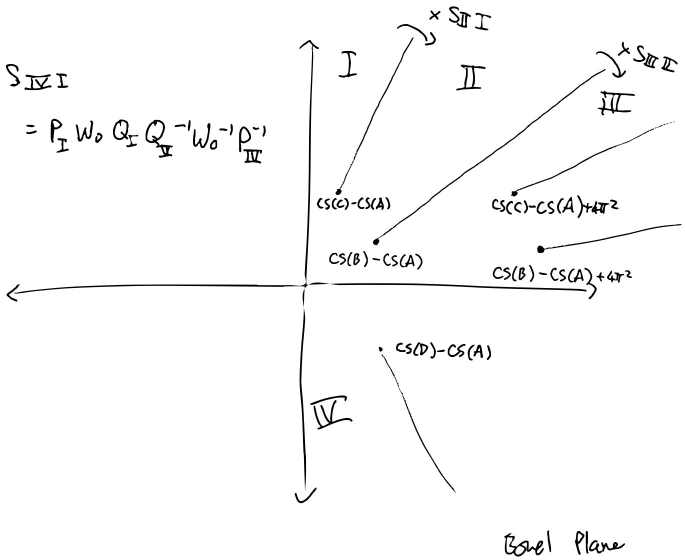
then the Stokes matrices are conjecturally given by

$$P_I(\tilde{q})W_0(\tilde{q})Q_I(\tilde{q})Q_{II}(\tilde{q})^{-1}W_0(\tilde{q})^{-1}P_{II}(\tilde{q})^{-1}\,.$$

Stoke automorphisms: Picture



Example: $4_1(1,2)$

For example, the Stokes indices for singularities in the Borel plane with positive imaginary part for $4_1(1,2)$ can be collected into generating series. The first few terms of the top corner of the full 8×8 matrix of Stokes indices can be computed to be

$$\begin{pmatrix} -q - 2q^2 - q^3 + 2q^4 + 6q^5 + \cdots & 1 + q - 2q^3 - 5q^4 - 7q^5 + \cdots \\ q^2 + 2q^3 + q^4 - q^5 + \cdots & -q - q^2 + q^4 + 5q^5 + \cdots \end{pmatrix}$$

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The closest singularities come from setting $q = 0$ which gives the matrix for the 7×7 part

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

Thanks!

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