Differential equations and deformations of QFT

Andrew Neitzke

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I learned most of what I know about this subject through joint work with Davide Gaiotto and Greg Moore, and subsequent joint work with Chris Beem, David Ben-Zvi, Mat Bullimore, Tudor Dimofte, David Dumas, Laura Fredrickson, Alba Grassi, Qianyu Hao, Lotte Hollands, Ali Shehper, Fei Yan.

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Such a defect generally has a moduli space C of chiral couplings. C is a complex analytic space.

There is a dictionary which connects this situation to (a family) of linear ODE defined on C, with meromorphic coefficients and a parameter ϵ. [Dorey, Dunning, Gaiotto, Jeong, Moore, N, Nekrasov, Shatashvili, Tateo, ...]

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$$\psi_f(z, \epsilon) = \exp\left(\epsilon^{-1} \int^z \phi(z')\, \mathrm{d}z'\right) = \exp\left(\epsilon^{-1} \int^z \sum a_n(z') \epsilon^n \, \mathrm{d}z'\right)$$

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Singularities of (the analytic continuation of) $\mathcal{B}\phi(z, \zeta)$ are responsible for Stokes phenomena: nonperturbative jumps of local solutions $\psi(z, \epsilon)$, important for the global analysis. [Ecalle, Kawai, Silverstone, Takei, Voros, ...]

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A singularity of $\mathcal{B}\psi(z,\zeta)$ at $\zeta = \zeta_0(z)$ corresponds to a BPS particle in the surface defect theory. The quantity $\zeta_0(z)$ is the "central charge" of the particle. (In particular, the mass of the particle is $M = |\zeta_0(z)|$.)

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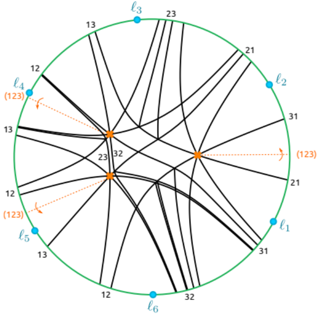
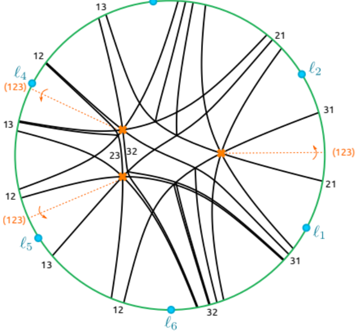
(BPS particles of the 4d theory without defect also show up as singularities on the ODE side: in Borel summation of Voros symbols / cluster coordinates.)

One can fix ϵ and look where Stokes phenomena occur in the parameter-space C: this gives the Stokes graph / spectral network [Gaiotto, Kawai, Moore, N, Takei, Voros, ...]

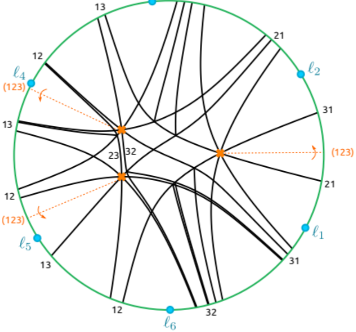


For example, this one appears in the (A_2, A_2) Argyres-Douglas theory, corresponding to the equation

$$\left(\epsilon^3 \partial_z^3 + \frac{1}{2}(-z^3 + 3z^2 + 2)\right)\psi(z) = 0$$



The spectral network is a useful tool for determining the BPS spectrum and for quantitative analysis of the ODE — eg computing Stokes data / monodromy.



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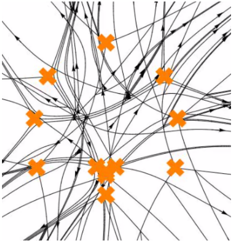
For the case above, one can make precise statements about the Stokes data, determining e.g. their asymptotic expansions around $\epsilon = 0$, and identifying them as solutions of integral equations. These are not theorems, mainly because Borel summability of the local solutions is not proven; but they can be tested numerically. [Dumas, N]

The complexity of the network poses practical challenges, especially in higher rank cases, eg for (A_3, A_7) one meets pictures like the one below:



They have resisted analysis so far.

(We'll return to this later in the talk.)

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One (slightly indirect) way to get it: study the vacuum Hilbert space of the defect theory on S^1 of radius R. This gives a complex vector bundle V_R over C, and tt^* geometry predicts V_R carries a family of flat connections, parameterized by $\zeta \in \mathbb{C}^\times$. Then take "conformal limit" $R \to 0$, $\zeta \to 0$, with $\epsilon = \zeta/R$ fixed. [Cecotti, Gaiotto, Moore, N, Vafa]

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But the ODE-QFT dictionary also arises more directly, upon studying an "S^1-equivariant" (Ω background) version of the 2d-4d system, where S^1 acts by rotation in the plane of the surface defect. Then ϵ is the equivariant parameter. [Jeong, Nekrasov, Shatashvili]

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There are similar structures in other dimensions, with different flavor. Right now we want the 2-dimensional version.

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Putting these structures together we get a holomorphic map $\varphi : TC \to \mathrm{End}(E)$, obeying $\varphi \wedge \varphi = 0$. This means E is naturally a Higgs bundle over C.

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We can average over the circle, defining

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Thus we have no multiplication in E_ϵ for $\epsilon \neq 0$ — looks like E_ϵ is just a (pointed) vector bundle over C.

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The two problems we just discussed can be made to cancel one another. If we perturb by $t\mathcal{O}^{(2)}$ *and also* turn on Ω background parameter ϵ, then in the perturbed theory the combination

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Said otherwise, the bundle E_ϵ does carry a natural ϵ-connection. (Rescale it by ϵ^{-1} to get an ordinary connection.)

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This point of view is close to ones pursued before from many different directions: particularly 2d TFT, Gromov-Witten theory. [Ben-Zvi, Dijkgraaf, Dubrovin, Getzler, Givental, Goodwillie, Kontsevich, Nadler, Nekrasov, Verlinde, Verlinde, Witten, ...]

To develop it precisely in our context, see it works beyond first order in ϵ, and see that we get exactly the expected ODEs, is work in progress [N, Shehper].

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It appears that, in parallel to the previous case, BPS particles in the field theory correspond to Stokes phenomena for the difference equation. [Alim, Beem, Cecotti, Dimofte, Gaiotto, Grassi, Hao, Hollands, N, Pasquetti, Tulli, Vafa, ...]

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There are Stokes phenomena along two rays in the ϵ-plane. These correspond directly to the two BPS particles of the theory.

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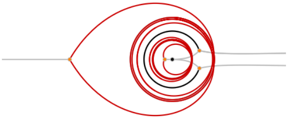
There are Stokes phenomena along infinitely many rays in the ϵ-plane. These correspond to the infinitely many Kaluza-Klein modes of the chiral field. [Beem-Dimofte-Pasquetti, Dimofte-Gaiotto-Gukov, Cecotti-Gaiotto-Vafa, Garoufalidis-Kashaev, Grassi-Hao-N, Alim-Hollands-Tulli, ...]

In 3d-5d systems there is an analog of spectral network, called exponential network [Banerjee, Eager, Longhi, Romo, Selmani, Walcher, ...] which governs the BPS spectrum; e.g. part of the network for a 3d defect in 5d $SU(2)$ super Yang-Mills is shown below:



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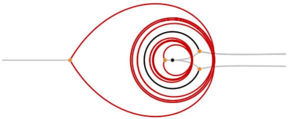

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This should be a useful clue toward development of the WKB method for difference equations. [Dingle, Kashani-Poor, ...]

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In some cases I want to suggest that it is really more natural to consider these, rather than the 1-dimensional ones.

We consider the 4d (A_2, A_1) Argyres-Douglas theory.

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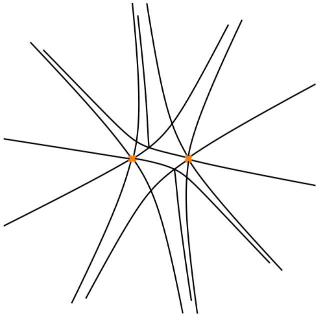
It admits a conformally invariant surface defect with chiral ring generated by $\mathcal{O}$ and σ, obeying $\sigma^3 + \mathcal{O} = 0$. Perturbing the surface defect by the operator σ, with coefficient y, this relation becomes

$$\sigma^3 + y^2 + \mathcal{O} = 0\,.$$

This chiral ring relation quantizes to an ODE in the y-plane,

$$(\epsilon^3 \partial_y^3 + y^2 + \langle \mathcal{O} \rangle)\, \psi = 0$$



One could study this equation by itself.

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Over this space we have a chiral ring

$$\sigma^3 + (z\sigma + y)^2 + \mathcal{O} = 0$$

which naturally embeds into $T^*\hat{C}$. Quantizing it gives a system of compatible equations

$$(\epsilon^3\partial_y^3 + (\epsilon z\partial_y + y)^2 + \langle\mathcal{O}\rangle)\psi = 0\,,$$

$$(\epsilon\partial_z - \frac{1}{2}\epsilon^2\partial_y^2)\psi = 0\,.$$

As we vary z, the equation in the y-plane undergoes iso-Stokes deformation (shown in accompanying notebook).

What is this good for?

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- As we saw before: going to large z seems to simplify the spectral network.
- In rank 2 theories, the geometry of Stokes phenomena has to do with strip decompositions of C. Simple pieces, simple transition functions from one piece to another ("half-translation surface".)

  In higher rank, this is not true: the local structure on C is complicated, given by many overlapping foliations. *But it becomes true again* if we use the higher-dimensional parameter space $\hat{C}$.

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These are "irrelevant" deformations in the language of the renormalization group flow (scaling dimension > 2). This is much more delicate than what we considered up to now: it's not clear *a priori* that theories obtained by these deformations really exist. They might depend on some additional choice of how to "UV complete" the theory.

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Wild guess: perhaps there is a UV completion corresponding to the 3-KdV hierarchy. This is a natural infinite family of iso-Stokes deformations of the equation in the y-plane, parameterized by higher "times" which should be naturally dual to combinations of the σ^n.

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It would not be surprising to see KdV hierarchy appear here: cf. its appearance in matrix models, topological strings, minimal string theories [Aganagic, Dijkgraaf, Douglas, Klemm, Marino, Moore, Vafa, ...]

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A class S theory admits a family of surface defects, parameterized by $y \in C$. The chiral ring has the form

$$\sigma^N + \sum \sigma^{N-i} \mathcal{O}_i(y) = 0$$

where each $\mathcal{O}_i(y)$ is a Coulomb branch operator of the 4d theory. [Alday, Drukker, Gaiotto, Gomis, Gukov, Moore, N, Okuda, Seiberg, Tachikawa, Teschner, Verlinde, ...]

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The deformation by σ shifts the parameter y. This is a marginal deformation (scaling dimension = 2).

But we also have available the irrelevant deformations by σ^k, for $k > 1$.

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I don't have a complete picture of how to think about these deformations. But let's simplify a bit by considering only the situation at the origin of the Coulomb branch of the 4d theory, i.e. set all $\mathcal{O}_i = 0$. In this case we only have $\sigma, \sigma^2, \dots, \sigma^{N-1}$.

These deformations should parameterize some $(N-1)$-dimensional extended moduli space $\hat{C}$ of surface defects, containing the original curve $C \subset \hat{C}$.

A candidate geometric description of the extended space $\hat{C}$: $\hat{C}$ parameterizes holomorphic Lagrangian subspaces of the $(N-1)$-th order infinitesimal neighborhood of T^*C.

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(Why? One heuristic: consider T^*C and a Lagrangian subspace $L \subset T^*C$, intersecting the zero section transversely. Then L determines a surface defect in the class S theory, via intersection of M5-branes in T^*C. The class S theory at the origin of the Coulomb branch only sees the expansion of L up to $(N-1)$-th order around C.)

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Recently, from a very different perspective, [Reid] studies this space $\hat{C}$ and a natural Higgs bundle on it, relates deformations of $\hat{C}$ to rank N *higher complex structures* on C [Fock, Thomas].

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We know that the moduli space of complex structures on C is a space of marginal (scaling dimension $= 4$) deformations of the 4d class S theory.

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Could it be that the moduli space of higher complex structures on C is a space of irrelevant (scaling dimension > 4) deformations of the 4d class S theory?

At least, these deformations do naturally produce higher Beltrami differentials [Shehper, N, Nekrasov, ...].

Thank you!