Instanton counting and q-deformation of Virasoro/W constraint

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1. Virasoro/W constraint for matrix model
2. Q-deformation of Virasoro/W constraint for instanton partition function
3. Geometry & representation theory aspects of q-Virasoro/W algebras

- Reference: https://arxiv.org/abs/2012.11711 (review)

1. Virasoro / W constraint for matrix model

Hermitian matrix model

$$Z(t) = \int dH \, e^{-\mathrm{Tr}\,V(H)}$$

$$\left(\text{potential} : V(x) = \sum_{n=1}^{\infty} \frac{t_n}{n} x^n \right)$$

Virasoro constraint

$$L_n \cdot Z(t) = 0 \quad \text{for} \quad n \geq -1$$

where $[L_n, L_m] = (n-m) L_{n+m}$

$$L_k \in \mathbb{C}[\![t_n, \tfrac{\partial}{\partial t_n}]\!]$$

Underlying symmetry : $H \to f(H) = \sum_{n=0}^{\infty} f_n \cdot H^n$

$$\left(\int dH \frac{\partial}{\partial H} (\ast) = 0 \right)$$

Generating current : $T(x) = \sum_{n \in \mathbb{Z}} L_n / x^{n+2}$

$$T(x) \cdot Z(t) = \sum_{n < -1} L_n \cdot Z(t) / x^{n+2}$$ (no negative powers in x)

$\Rightarrow$ Polynomiality of T-op. average

$$\langle T(x) \rangle = x^{\#} + \cdots$$

Spectral curve : $W(x)^2 + \frac{1}{N} W'(x) = V'(x) W(x) - P(x)$

u^2

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$\begin{pmatrix} W(x): \text{resolvent} \\ P(x): \text{polynom. fn.} \end{pmatrix}$ $\Updownarrow$

$$\underbrace{\overbrace{\boxed{\left(W(x) - \frac{V'(x)}{2}\right)^2}}^{y^2} + \left(\frac{1}{N} W'(x)\right)}_{\langle T(x) \rangle} = \frac{V'(x)^2}{4} - P(x)$$

$$\Rightarrow \Sigma = \{ (x, y) \mid y^{\boxed{2}} = \underbrace{\frac{V'(x)^2}{4} - P(x)}_{\text{polynom.}} \}$$

$2 \to r$

W - constraint : multi - matrix model

$$Z = \int \prod_{\substack{(a,i) \\ \neq (b,j)}} (x_{a,i} - x_{b,j})^{C_{ab}} \times (\text{potentials})$$

C_{ab}: Cartan matrix of alg. $\mathfrak{g}$

$\Rightarrow$ $W(\mathfrak{g})$ constraint $\to W(^L\mathfrak{g})$

$$\begin{cases} \mathfrak{g} = sl_2 \quad (A_1) \quad C = (2) \quad W(sl_2) = \text{Virasoro} \\ \mathfrak{g} = sl_r \quad (A_{r-1}) \quad C = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \quad W(sl_r) = W_r \end{cases}$$

2. q-deformation of Virasoro/W constraint for instanton partition function

4d N=2 G-YM theory partition fn.

$$Z_G = \int DA \, e^{-S_{YM}[A]}$$

$$\to \sum_{k=0}^{\infty} \underbrace{q^k Z_{k,G}}_{\text{instanton}} \left(+ \underbrace{\bar{q}^k \overline{Z_{k,G}}}_{\text{anti-inst.}} \right)$$

where $Z_{k,G} = \text{vol.}\,(\underbrace{\mathcal{M}_{k,G}}_{\substack{\text{mod. sp. of } k\text{-instanton sol.} \\ \text{in G-YM theory}}}) = \int_{\mathcal{M}_{k,G}} 1$

$q = e^{2\pi i \tau}$, τ : cmplx coupling

e.g. (S)U(n) ; Localization $Z_{U(n)} = \sum_{\lambda} q^{|\lambda|} Z_\lambda$

e.g. $(5)U(n)$, localization $\subset U(n)$ $\quad \underbrace{\sum_{\lambda}}_{\text{n-tuple partition}} + \underbrace{\sum_{\lambda}}_{\text{discretized Vandermonde det}}$

"Potential term" : $Z(t) = \sum_{\lambda} q^{|\lambda|} Z_{\lambda} \cdot \underbrace{\boxed{Z_{\lambda}^{\mathrm{pot}}(t)}}_{Z_{\lambda}(t)}$

(generating fn of chiral rings)

$$Z_k = \int_{\mathcal{M}_{k,G}} 1 \;\; \exp\left(\sum_{n=1}^{\infty} t_n \cdot \mathrm{ch}\, \mathcal{Y}^{[n]}\right)$$

<u>Prop.</u> : T-op. average is a polynom.

$$\langle T(x) \rangle = x^n + \cdots \quad (\text{for } G = U(n))$$

where $T(x) = \sum_{n \in \mathbb{Z}} T_n / x^n$ is the gen. current of

q-Virasoro alg. Vir_{q_1, q_2}

$$T_n \in \mathbb{C}[\![t_m, \tfrac{\partial}{\partial t_m}]\!]$$

Average : $\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{\lambda} q^{|\lambda|} Z_{\lambda} \cdot \mathcal{O}_{\lambda}$

Underlying (Str. / symmetry : discrete variable change.

$$\left(\mathcal{M}_{k,G} \longrightarrow \mathcal{M}_{k+1,G} \right)$$
$$= \{\cdots\}/U(k) \qquad\qquad = \{\cdots\}/U(k+1)$$

- q-W constraint : quiver gauge theory ($\simeq$ multi-matrix)

<u>Prop.</u> (quiver W-alg.) : For Γ-quiver gauge th.,

$$\langle T_i(x) \rangle = x^{n_i} + \cdots = \prod_{k=1}^{n_i} (x - a_k)$$

where $\{T_i(x)\}$ are gen. currents of $W_{q_1 q_2}(\Gamma)$

$k=1$

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e.g. $\Gamma = A_{r-1}$: $\underset{1}{\overset{U(n_1)}{\circ}} - \underset{2}{\overset{U(n_2)}{\circ}} - - \cdots - \underset{r-1}{\overset{U(n_{r-1})}{\circ}}$

$$W_{q_1 q_2}(A_{r-1}) = q\text{-}W_r \text{ alg.}$$

$$(q_1, q_2) = (e^{\epsilon_1}, e^{\epsilon_2}) \in U(1)^2 \subset U(2)$$

3. Geometry & rep. theory aspects.

eg. $\Gamma = A_3$: $\circ - \circ - \circ \Rightarrow sl_4$

Free field realization of $\{T_i(x)\}$: $\{Y_i(x)\}_{i=1,2,3}$

$T_{\square}$ 4 dim $SL(4)$
$$T_1(x) = Y_1(x) + \frac{Y_2}{Y_1} + \frac{Y_3}{Y_2} + \frac{1}{Y_3}$$

$T_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ 6 dim $SL(4)$
$$T_2(x) = Y_2(x) + \frac{Y_1 Y_3}{Y_2} + \frac{Y_1}{Y_3} + \frac{Y_3}{Y_1} + \frac{Y_2}{Y_1 Y_3} + \frac{1}{Y_2}$$

$T_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ 4 dim $SL(4)$
$$T_3(x) = Y_3(x) + \frac{Y_2}{Y_3} + \frac{Y_1}{Y_2} + \frac{1}{Y_1}$$

<u>Prop.</u> : $\{T_i(x)\}$ are given by the qq-character of the fund. reps. of Γ-quiver.

$\Rightarrow$ Quadratic rel's

$$T_{\square} \wedge T_{\square} \to T_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, \quad T_{\square} \wedge T_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \to T_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$$

"$[T_{\square}, T_{\square}] \approx T_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$"