Mining Perturbation Theory: Resurgence-Inspired Extrapolation and Analytic Continuation

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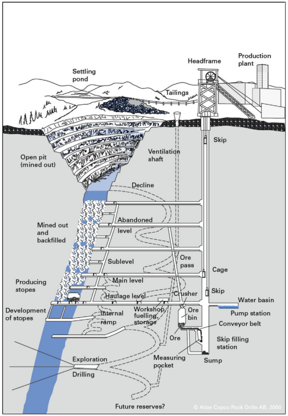
September 6, 2022

O.Costin & GD: 1904.11593, 2003.07451, 2009.01962, 2108.01145
GD & Z.Harris: 2101.10409 + 2209.xxxx
G.Basar, GD, Z.Yin 2112.14269

[DOE Division of High Energy Physics]

Statement of the Problem

- a common problem in applications: we can only compute a <u>finite number</u> (often a small number ~ 10) of coefficients of an expansion of a function about some special parameter point, and we wish to learn about the behaviour near <u>another point</u> (possibly very distant)



- *inverse approximation theory*

- what is <u>the best</u> way to "mine" this perturbative data ?

Inspiration from Resurgence

- there is growing evidence for resurgent behaviour of physical functions
- resurgence suggests that expansions about different points are (generically) quantitatively related
- idea: take advantage of this to develop optimal, and practical near-optimal, extrapolation & analytic continuation methods
- in many applications the Borel plane has <u>structure</u>, because it has physical meaning (it is the non-perturbative physics)
- strategy: systematically reconstruct the behaviour near the singularities as precisely as possible
- basic toolkit:

  (conformal & uniformizing maps) + (Padé methods)
- rigorous proofs for resurgent functions

Physical Motivation: Resurgence and Quantum Field Theory

- resurgence suggests deep connections between perturbative and non-perturbative features of QFT

- weak-coupling $\leftrightarrow$ strong-coupling; high temperature $\leftrightarrow$ low temperature; adiabatic $\leftrightarrow$ non-adiabatic; Euclidean QFT $\leftrightarrow$ Minkowski QFT; magnetic field $\leftrightarrow$ electric field; ...

- QCD phase diagram ("sign problem")

- QED at high intensity (*terra incognita*: DESY and SLAC experimental proposals)

- QED, QCD β functions: 5th order perturbation theory

- electron (g-2): 5th order perturbation theory (muon?)

- $\phi_4^4(N)$: 7^{th} order exact; 11^{th} order Hepp bounds; $\sim 20^{th}$?? order; $\phi_6^3(d_{abc})$: 5^{th} order exact (Schnetz, Panzer, Borinsky, Gracey, ...)

- ... the Amplitude program is making rapid progress

Generic Dominant Bender-Wu-Lipatov Asymptotics

- generic leading behaviour in many applications

$$f(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^{n+1}} \quad , \quad x \to +\infty \quad ; \quad a_n \sim \mathcal{S}\,(-1)^n \frac{\Gamma(n-\alpha)}{b^n} \quad , \quad n \to \infty$$

- incomplete gamma function:

$$F(x) = x^{-1-\alpha}\, e^x\, \Gamma(1+\alpha, x)$$

- exact non-perturbative connection formula

$$F(e^{i\pi}\, x) - F(e^{-i\pi}\, x) = \frac{-2\pi i}{\Gamma(-\alpha)}\, \frac{e^{-x}}{x^{1+\alpha}}$$

- Borel representation

$$F(x) = \int_0^\infty e^{-px}(1+p)^\alpha \qquad , \qquad \mathcal{B}(p) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n-\alpha)}{\Gamma(-\alpha)\, n!}\, p^n$$

- Q: reconstruct $\mathcal{B}(p)$ from a finite number of terms?

Quantifying Padé-Borel & Padé-Conformal-Borel (Costin, GD: 2003.07451)

- exact Padé-Borel: in terms of Jacobi polynomials

$$
\begin{aligned}
\mathrm{PB}_{[N,N]}(p;\alpha) &= \frac{P_N^{(\alpha,-\alpha)}\left(1+\frac{2}{p}\right)}{P_N^{(-\alpha,\alpha)}\left(1+\frac{2}{p}\right)} \\
&\sim \frac{I_\alpha\left(\left(N+\frac{1}{2}\right)\ln\left[\frac{\sqrt{1+p}+1}{\sqrt{1+p}-1}\right]\right)}{I_{-\alpha}\left(\left(N+\frac{1}{2}\right)\ln\left[\frac{\sqrt{1+p}+1}{\sqrt{1+p}-1}\right]\right)} \\
&\to p^\alpha\left(\frac{N^2}{p}\right)^\alpha \quad , \quad p\to\infty
\end{aligned}
$$

$\Rightarrow$ accuracy down to $x_{\min}\sim\frac{1}{N^2}$ (already impressive)

- note: Padé generates its own conformal map (Szegö; Stahl)

- but Padé is not accurate near the branch point or cut $\Rightarrow$ limited access to the connection formula

Quantifying Padé-Borel & Padé-Conformal-Borel (Costin, GD: 2003.07451)

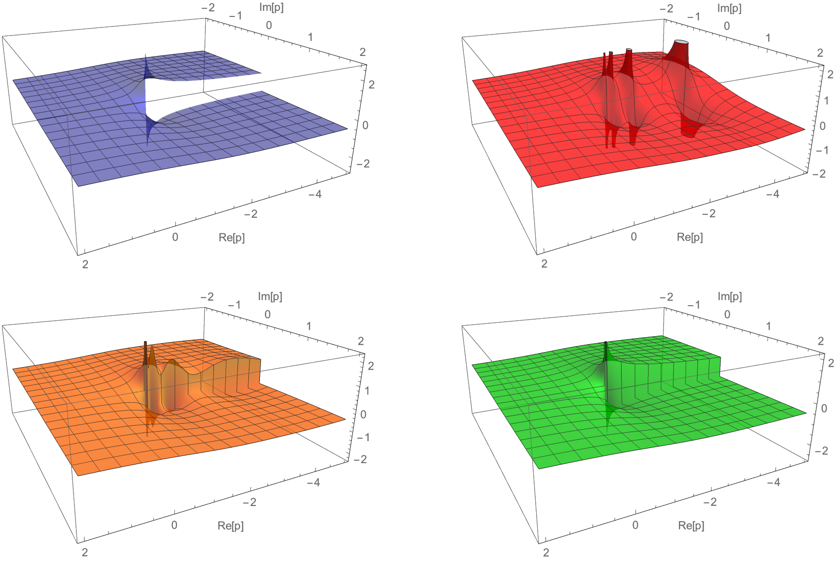
- conformal map: $z=\frac{\sqrt{1+p}-1}{\sqrt{1+p}+1} \quad \longleftrightarrow \quad p=\frac{4z}{(1-z)^2}$
- exact Padé-Conformal-Borel: Jacobi polynomials again

$$
\begin{aligned}
\mathrm{PCB}_{[N,N]}(p;\alpha) &= \frac{P_N^{(2\alpha,-2\alpha)}\left(\frac{\sqrt{1+p}+1}{\sqrt{1+p}-1}\right)}{P_N^{(-2\alpha,2\alpha)}\left(\frac{\sqrt{1+p}+1}{\sqrt{1+p}-1}\right)} \\
&\sim \frac{I_{2\alpha}\left(\left(N+\frac{1}{2}\right)\ln\left[\frac{((1+p)^{\frac{1}{4}}+1)^2}{(\sqrt{1+p}-1)}\right]\right)}{I_{-2\alpha}\left(\left(N+\frac{1}{2}\right)\ln\left[\frac{((1+p)^{\frac{1}{4}}+1)^2}{(\sqrt{1+p}-1)}\right]\right)} \\
&\to p^{\alpha}\left(\frac{N^4}{p}\right)^{\alpha} \quad , \quad p\to\infty
\end{aligned}
$$

$\Rightarrow$ accuracy down to $x_{\min}\sim\frac{1}{N^4}$ (much better)

- Padé-Conformal is also accurate near the branch point or cut, because the Jacobi poles are on the next sheet

Padé-Conformal-Borel: 10-term approximation to $(1+p)^{-1/3}$

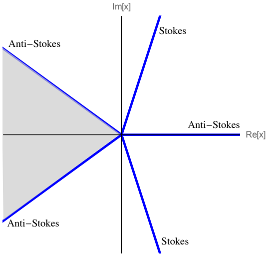


message: simple steps can lead to significant improvement

Analytic Continuation of Painlevé I *tritronquée* (Costin, GD: 1904.11593)

- Painlevé I: $y''(x) = 6\, y^2(x) - x$
- series expansion as $x \to +\infty$

$$y(x) \sim -\sqrt{\frac{x}{6}}\left(1 + \sum_{n=1}^{\infty} c_n \left(\frac{30}{(24\,x)^{5/4}}\right)^{2n}\right)$$



- 5-fold symmetry: $y(x) \approx \sqrt{x}\,\mathcal{P}\left(\frac{4}{5}x^{5/4}; \{2, g_3\}\right)$ (Boutroux)
- *tritronquée*: poles only in $\frac{2\pi}{5}$ wedge (Dubrovin et al)

$$y(x) \approx \frac{1}{(x - x_{\rm pole})^2} + \frac{x_{\rm pole}}{10}(x - x_{\rm pole})^2 + \frac{1}{6}(x - x_{\rm pole})^3$$
$$+ h_{\rm pole}(x - x_{\rm pole})^4 + \frac{x_{\rm pole}^2}{300}(x - x_{\rm pole})^6 + \ldots$$

- Q: does the expansion as $x \to +\infty$ "know" this?

Metamorphosis: Asymptotic Series to Meromorphic Function

$$y(x) = \frac{1}{(x - {\color{red}x_{\rm pole}})^2} + \frac{x_{\rm pole}}{10}(x - x_{\rm pole})^2 + \frac{1}{6}(x - x_{\rm pole})^3 + {\color{red}h_{\rm pole}}(x - x_{\rm pole})^4 + \frac{x_{\rm pole}^2}{300}(x - x_{\rm pole})^6 + \dots$$

- Padé-Conformal extrapolation of $y(x)$ near 1st pole:

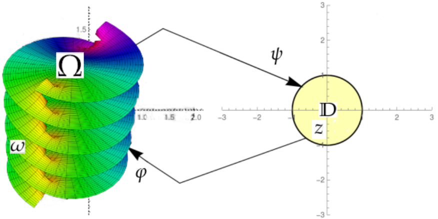
$$\begin{aligned} y(x) \approx\ & \frac{0.9999999999999999999999999999999999997886}{(x - x_1)^2} \\ & + 3.5 \times 10^{-35} - 2.4 \times 10^{-34}(x - x_1) \\ & {\color{red}-0.238416876956881663929914585244923803}(x - x_1)^2 \\ & + 0.166666666666666666666666666666657864(x - x_1)^3 \\ & {\color{red}-0.06213573922617764089649014164005140}(x - x_1)^4 \\ & + 4 \times 10^{-31}(x - x_1)^5 \\ & + 0.0189475357392909503157755851627665(x - x_1)^6 + \dots \end{aligned}$$

- approx. 30 digit precision for x_1 and h_1
- exact connection formulas satisfied to high precision

Optimality?

but we can do even better than this ...

Optimality Theorem (O.Costin, GD: 2009.01962)



• unif. map $\psi(0) = 0$, $\psi'(0) > 0$
• within class of functions with common Riemann surface Ω, the optimal reconstruction procedure is:

1. uniformization map, $\psi : \Omega \to \mathbb{D}$, of truncated expansion
2. re-expand in z inside $\mathbb{D}$ to the same (!) order
3. map back to Ω

- explicit maps known for $\hat{\mathbb{C}} \setminus \{\omega_1, \omega_2, \omega_3\}$ + symmetries
- e.g. uniformization of Borel plane for PI-PV *tronquée*:

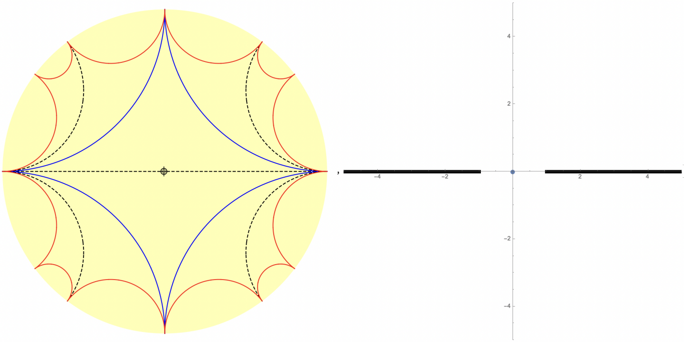
$$\varphi = \frac{1}{2\pi i} \ln\left(1 - q^{-1}\right)$$

- note: many applications have <u>few dominant</u> singularities

Exploring Different Riemann Sheets

- uniformization of $\hat{\mathbb{C}} \setminus \{-1, 1, \infty\}$: modular λ function

$$w(z) = -1 + 2\lambda\left(i\left(\frac{1+i\,z}{1-i\,z}\right)\right)$$



- interactive Mathematica file for uniformization

- physics application: crossing Riemann sheets near critical points (see later)

Near-Optimal Numerical "Exploratory" Methods

- Singularity Elimination method:
  1. super-precise probe of vicinity of an isolated singularity

$$f(\omega) \sim A(\omega)\,(\omega - \omega_c)^{\beta} + B(\omega) \qquad , \quad \omega \to \omega_c$$

  2. linear transformation (fractional deriv.) converts index to $\frac{1}{2}$
  3. conformal map (e.g. $\omega \to 2\omega_c\omega - \omega^2$) removes singularity

- high precision access to singularity location, index and fluctuations

- access to higher sheets (see later)

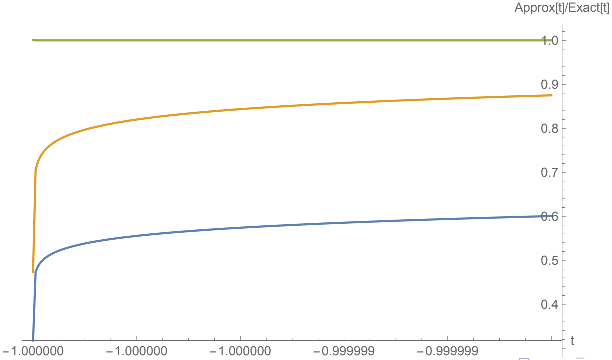
- once removed, can proceed to another singularity, etc ...

Probing the Neighborhood of an Isolated Singularity

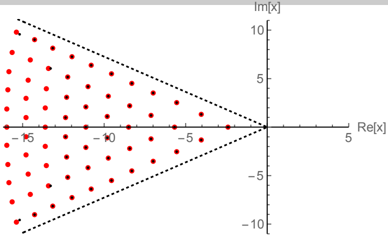
- exponential distortion near a uniformized singularity
- e.g. elliptic nome function (20 terms)

$$z = \exp\left[-\pi\,\frac{\mathbb{K}(1-t)}{\mathbb{K}(t)}\right] \quad \longleftrightarrow \quad t = \varphi(z) = 16z - 128z^2 + 704z^3 + \ldots$$

- $z \approx 0.9 \quad \longleftrightarrow \quad t \approx -1 + 10^{-40}$



Fine Structure in the Tritronquée Pole Region



- uniformization → 66 poles
- excellent agreement with trans-asymptotics
- high-precision Mittag-Leffler "sum rules" for “lattice” of poles
- *cf.* Eisenstein series

$$y(x; x_{\text{pole}}, h_{\text{pole}}) = \frac{1}{(x - x_{\text{pole}})^2} + \sum_{\Omega \neq x_{\text{pole}}} \left(\frac{1}{(x-\Omega)^2} - \frac{1}{\Omega^2} \right)$$

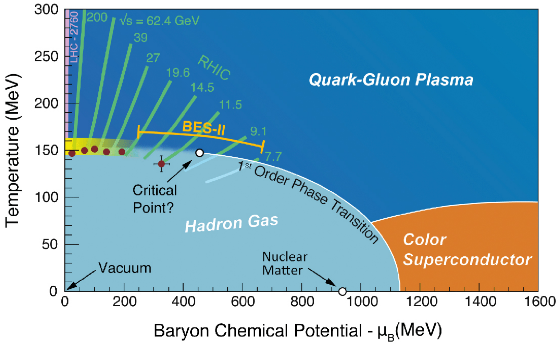
$$\sum_{\Omega \neq x_{\text{pole}}} \frac{1}{(\Omega - x_{\text{pole}})^4} = \frac{x_{\text{pole}}}{30} \quad ; \quad \sum_{\Omega \neq x_{\text{pole}}} \frac{1}{(\Omega - x_{\text{pole}})^5} = \frac{1}{24}$$

$$\sum_{\Omega \neq x_{\text{pole}}} \frac{1}{(\Omega - x_{\text{pole}})^6} = \frac{h_{\text{pole}}}{5} \quad ; \quad \sum_{\Omega \neq x_{\text{pole}}} \frac{1}{(\Omega - x_{\text{pole}})^7} = 0$$

$$\sum_{\Omega \neq x_{\text{pole}}} \frac{1}{(\Omega - x_{\text{pole}})^8} = \frac{x_{\text{pole}}^2}{2100} \quad ; \quad \ldots \text{ satisfied to high precision}$$

- also higher sheets ... recall Ovidiu’s seminar + next example

Application: QCD Phase Diagram



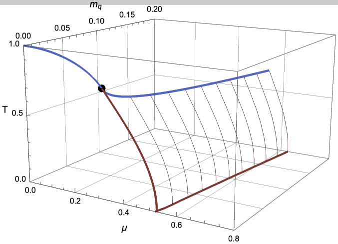
- major problem in QFT: finite density at strong coupling
- common (uncontrolled) strategy: use imaginary chemical potential & "analytically continue" truncated Taylor series
- search for QCD critical point: Ising universality class
- need access to higher sheets: how?

Chiral Random Matrix Model for the QCD Phase Diagram

- chiral random matrix model

(Halasz et al 1998; Stephanov 2004; ...)

- partition function $Z(T,\mu)$ has complex Lee-Yang zeros at finite N

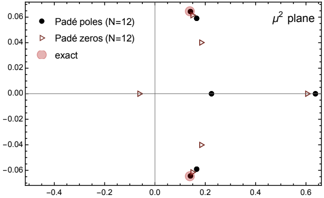


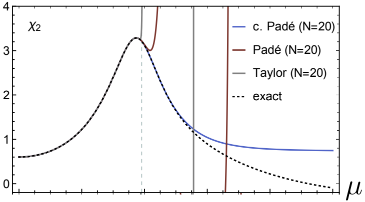
$$Z(T,\mu) = \int \mathcal{D}\Phi e^{-N\mathrm{Tr}(\Phi\Phi^\dagger)} \det{}^{\frac{N}{2}} \begin{pmatrix} \Phi + m_q & \mu + iT \\ \mu + iT & \Phi^\dagger + m_q \end{pmatrix} \det{}^{\frac{N}{2}} \begin{pmatrix} \Phi + m_q & \mu - iT \\ \mu - iT & \Phi^\dagger + m_q \end{pmatrix}$$

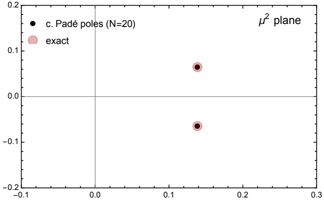
- pressure: $P(T,\mu) = \lim_{N\to\infty} \frac{1}{N} \ln Z(T,\mu)$
- susceptibilities: $\chi_2 := \frac{\partial^2 P(T,\mu)}{\partial\mu^2}$, $\chi_4 := \frac{\partial^4 P(T,\mu)}{\partial\mu^4}$
- small μ expansions: $\chi_k(T,\mu) = \sum_{n=0}^{N} c_n^{(k)}(T)\, \mu^{2n}$

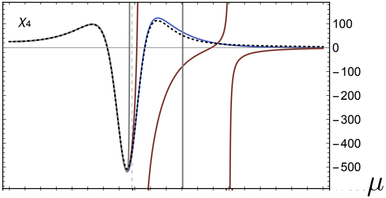
Uniformizing Lee-Yang Zeros (Basar, GD, Yin 2112.14269)

- mimic QCD: finite-order μ^2 expansion of susceptibilities
- Padé singularities are the (complex) Lee-Yang zeros
- with conformal map we can extrapolate beyond $|\mu_{LY}|$







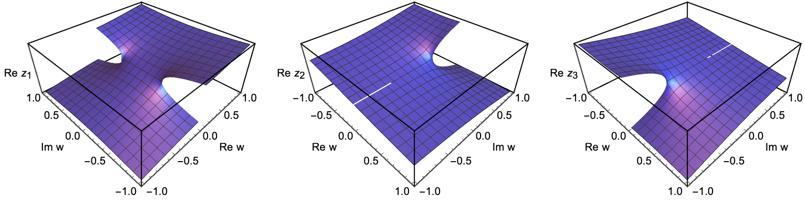


Equation of State: Extrapolating Between Riemann Sheets

- 3d Ising universality class (Stephanov, ...)

$$\text{effective potential:} \quad \Omega = -h\,M + \frac{r}{2}M^2 + \frac{1}{4}M^4$$

- near critical point: scaling $w := h r^{-\beta\delta}$, $z := M r^{-\beta}$
- mean field ($\beta = \frac{1}{2}$, $\delta = 3$): $\frac{\partial \Omega}{\partial M} = 0 \Rightarrow \boxed{w = z + z^3}$
- three sheets: $z_1(w)$ = high T sheet; $z_2(w)$ = low T sheet; $[z_3(w) = -z_2(-w)]$

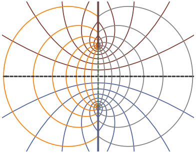


High T Equation of State: Extrapolation on First Riemann Sheet

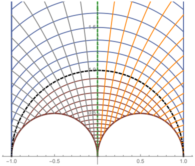
- high T expansion: $z_1(w) = w - w^3 + 3w^5 - 12w^7 + \ldots$
- uniformization: $\lambda(\tau)$= modular lambda function

$$w(\tau) = i(-1 + 2\lambda(\tau)) \qquad ; \qquad \tau(\zeta) = i\left(\frac{1 + i\,\zeta}{1 - i\,\zeta}\right)$$

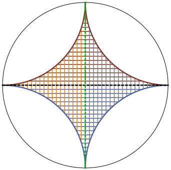
high T: w plane



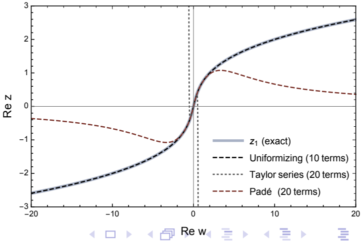
high T: τ plane



high T: ζ plane

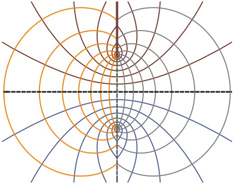


- extrapolation based on 10 terms of high T expansion

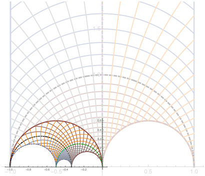


Equation of State: Continuation to Low Temperature Sheet

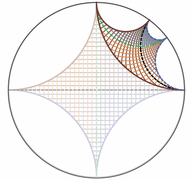
low T: w plane



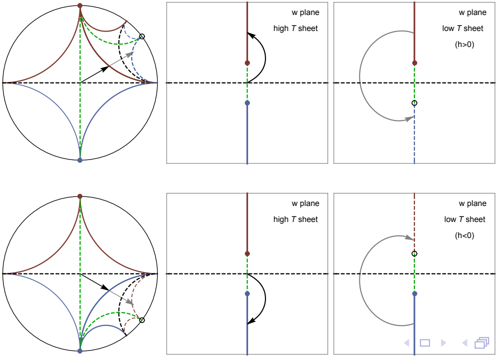
low T: τ plane



low T: ζ plane

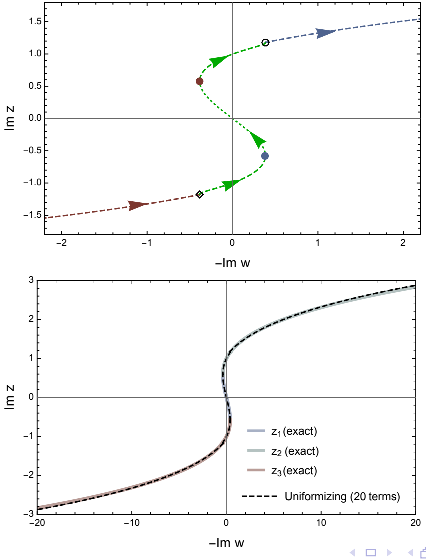


- traverse between sheets by moving in unit ζ disk



Equation of State: Continuation to Low T Riemann Sheet

- Padé in ζ plane $\rightarrow$ reconstruct function on low T sheet



Application: Heisenberg-Euler Effective Action

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{\hbar c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}.$$

$\Big(\mathfrak{E}, \mathfrak{B}$ Kraft auf das Elektron.
$|\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\,\frac{e}{(e^2/mc^2)^2}$ = „Kritische Feldstärke“.$\Big)$

- the first (non-perturbative) QFT computation
- paradigm of "effective field theory" (non-linear)
- compute: $\ln\det(\not{D}+m) \quad , \quad \not{D} := \not{\partial} + e\not{A}$
- at higher perturbative order, and for inhomogeneous background fields, closed formulas are not known

Application: Heisenberg-Euler Effective Action(s)

- intense (strongly inhomogeneous) fields: *terra incognita*
- 1-loop: $\ln\det\left(\not{D}+m\right) \qquad \not{D} := \not{\partial} + e\not{A}$

1. Fredholm Determinant (Matthews/Salam, Schwinger, ...)
2. Worldline Path Integral (Feynman, Nambu,...)
3. WKB approximation (Keldysh, Brézin/Itzykson, Popov/Marinov, ...)

- loop expansion: $\mathcal{L}\left(\alpha, \frac{eF}{m^2}\right) \sim \sum_{l=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^l \mathcal{L}^{(l)}\left(\frac{eF}{m^2}\right)$

- weak-field all-orders conjecture (Affleck/Alvarez/Manton; Ritus)

$$\mathrm{Im}\left[\mathcal{L}\left(\alpha, \frac{eE}{m^2}\right)\right] \sim \frac{\alpha E^2}{2\pi^2}\, e^{\pi\,\alpha}\, e^{-\pi\, m^2/(eE)} + \ldots$$

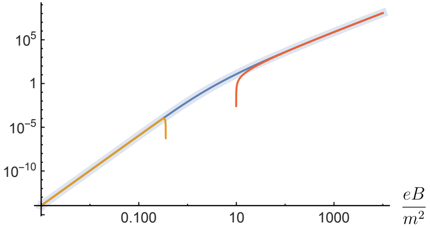
- Ritus-Narozhny Conjecture: QED expansion parameter in constant-crossed field, $\chi \gg 1$, is $\alpha\,\chi^{2/3}$, $\left(\chi \equiv \frac{e\sqrt{(F_{\mu\nu}p^\nu)^2}}{m^3}\right)$

Extrapolating Heisenberg-Euler Florio 1911.03489, GD/Harris 2101.10409

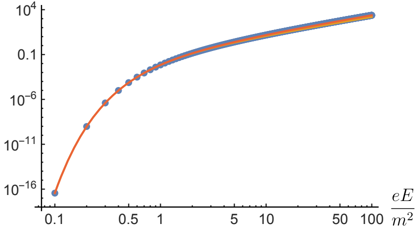
$$\begin{aligned}
\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) &= -\frac{B^2}{2}\int_0^\infty \frac{dt}{t^2}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right) e^{-m^2 t/(eB)} \\
&\sim \frac{B^2}{\pi^2}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}}\zeta(2n+4)\left(\frac{eB}{m^2}\right)^{2n} \quad , \quad eB \ll m^2 \\
&\sim \frac{1}{3}\cdot\frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma + \frac{6}{\pi^2}\zeta'(2)\right) + \ldots \quad , \quad eB \gg m^2
\end{aligned}$$

- small B → large B; small B → large E (from 10 terms!)

$\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right)$



$\text{Im}\,\mathcal{L}^{(1)}\left(\frac{eE}{m^2}\right)$



- exponentially suppressed terms are also accessible
- also at 2 loop (no Borel representation known) (GD/Harris 2101.10409)

Inhomogeneous Background Fields (GD & Z. Harris, to appear)

- "parametric resurgence": perturbative coefficients depend on the inhomogeneity parameter(s)

- precision tests for soluble cases (Narozhnyi/Nikishov, Popov, ...)

$$B(x) = B\,\mathrm{sech}^2\left(x/\lambda\right) \qquad E(t) = E\,\mathrm{sech}^2\left(\omega t\right)$$

- reduction to <u>single Borel integral</u> (Cangemi-GD-D'Hoker; GD-Hall)

- analytic continuations: $B^2 \mapsto -E^2$, $\lambda^2 \mapsto -1/\omega^2$

- Keldysh inhomogeneity parameter

$$\gamma = \frac{\ell_B^2}{\lambda_C \lambda} = \frac{m}{eB\lambda} \mapsto \frac{m\omega}{eE}$$

- WKB approximation: (Popov, ...)

$$\mathrm{Im}\left[S(E,\omega)\right]_{\mathrm{WKB}} \sim L^3 \frac{m^4}{8\pi^3\omega}\left(\frac{eE}{m^2}\right)^{5/2}(1+\gamma^2)^{5/4}\exp\left[-\frac{\pi m^2}{eE}\frac{2}{\sqrt{1+\gamma^2}+1}\right]$$

- truncated weak B field expansion

$$\frac{S(B,\lambda)}{L^2\lambda T} = \frac{m^4}{\pi^2}\sum_{n=0}^{N} a_n(\gamma)\left(\frac{B}{m^2}\right)^{2n+4}$$

- $a_n(\gamma)$: polynomial in inhomogeneity parameter $\gamma = \frac{m}{eB\lambda}$

- three fundamental Borel branch points can be seen in the large order growth of the perturbative coefficients $a_n(\gamma)$

$$\pm t_1 = \frac{\pm i}{\sqrt{1+\gamma^2}+1}, \quad \pm t_2 = \frac{\mp i}{\sqrt{1+\gamma^2}-1}, \quad \pm t_3 = \frac{\pm i}{\gamma}$$

$\Rightarrow$ three independent trans-series non-perturbative factors

$$\exp\left[\frac{-2\frac{\pi m^2}{eE}}{\sqrt{1+\gamma^2}+1}\right] \quad , \quad \exp\left[\frac{-2\frac{\pi m^2}{eE}}{\sqrt{1+\gamma^2}-1}\right] \quad , \quad \exp\left[\frac{-2\frac{\pi m^2}{eE}}{\gamma}\right]$$

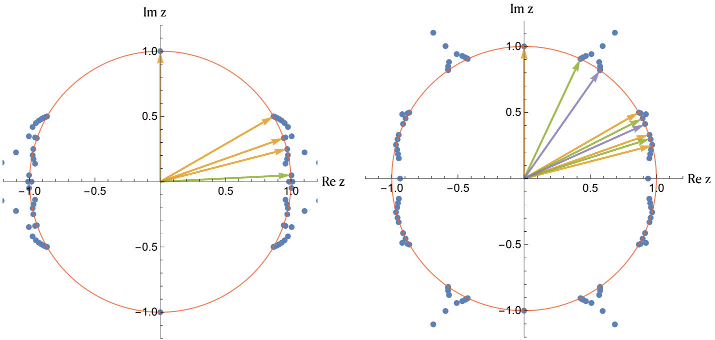
- subleading Borel singularities become important for inhomogeneous fields (large γ)

Borel Singularities for Inhomogeneous Background Fields

- the 3 Borel singularities are *collinear*, so Padé-Borel must be supplemented by a conformal map: Padé-Conformal-Borel

- Padé poles from expansion in the conformal disk

$\gamma = 10^{-1}$ $\gamma = 10$



- small γ: leading singularity (+ repetitions) dominate

- large γ: all singularities become relevant

Resurgence for Inhomogeneous Background Fields

- large order growth of perturbative coefficients

$$a_n(\gamma) \sim (-1)^n \Gamma(2n + \tfrac{3}{2}) \frac{3\sqrt{2\pi}}{|t_1|^{2n+3/2}} (1+\gamma^2)^{5/4}$$

$$\times \left[1 - \frac{5}{4} \frac{(1 - \frac{3}{4}\gamma^2)}{\sqrt{1+\gamma^2}} \frac{2|t_1|}{(2n + \frac{1}{2})} + \frac{105}{32} \frac{(1 + \frac{1}{4}\gamma^2)^2}{(1+\gamma^2)} \frac{(2|t_1|)^2}{(2n+\frac{1}{2})(2n-\frac{1}{2})} + \dots \right]$$

- fluctuations about leading worldline instanton:

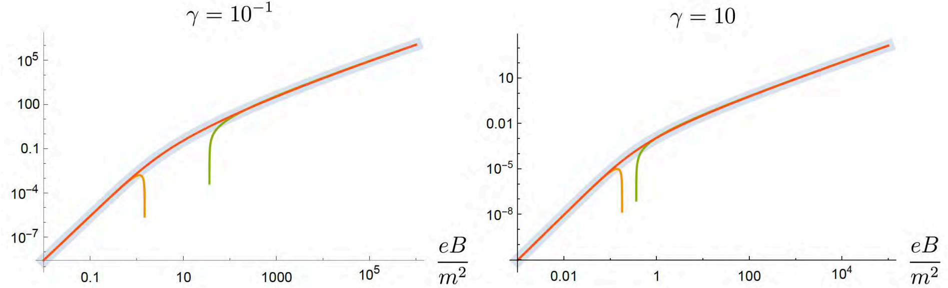
$$\frac{\mathrm{Im} S(E,\omega)}{L^2 T/\omega} \sim \frac{m^4}{8\pi^3} \left(\frac{E}{m^2}\right)^{5/2} (1+\gamma^2)^{5/4} \exp\left(-\frac{\pi m^2}{E} \frac{2}{\sqrt{1+\gamma^2}+1} \right)$$

$$\times \left[1 - \frac{5}{4} \frac{(1 - \frac{3}{4}\gamma^2)}{\sqrt{1+\gamma^2}} \left(\frac{E}{\pi m^2}\right) + \frac{105}{32} \frac{(1 + \frac{1}{4}\gamma^2)^2}{(1+\gamma^2)} \left(\frac{E}{\pi m^2}\right)^2 + \dots \right]$$

- also: other Borel singularities & all multi-instantons

Resurgent Extrapolation for Inhomogeneous Background Fields

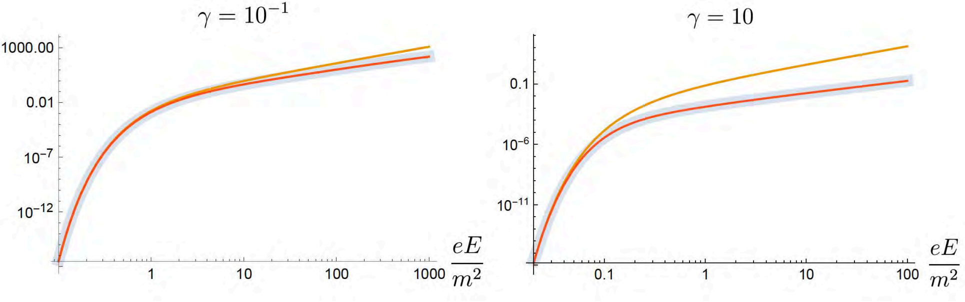
- analytic continuation: weak B field to strong B field
- with just 15 perturbative input terms



- accurate agreement over many orders of magnitude
- agrees with strong field limit (for all γ, even large γ)
- far superior to locally-constant-field approximation

Resurgent Extrapolation for Inhomogeneous Background Fields

- analytic continuation: <u>weak B field to strong E field</u>
- with just 15 perturbative input terms



- accurate agreement over many orders of magnitude
- agrees with strong field limit
- far superior to WKB or locally-constant-field approximation

Conclusions

- resurgent extrapolation: strong-field and non-perturbative and non-adiabatic information can be decoded efficiently from relatively modest amounts of perturbative data
- conformal and uniformizing maps, even for just leading singularities, lead to dramatic improvements of precision
- Painlevé I *tritronquée*: Stokes transition into pole region
- Chiral matrix model: extrapolation between sheets
- Heisenberg-Euler: resurgence & precise continuations
- higher loops? resummations? general inhomogeneities?
- effect of noisy coefficients (Costin, GD, Meynig: 2208.02410)
- many other potential applications in QFT

Extrapolating Series with Noisy Coefficients (Costin, GD, Meynig, 2208.02410)

- expansion coefficients may be known to finite precision

$$f(\omega) := \sum_{k=0}^{m} f_k\, \omega^k \quad \to \quad f(\omega) + \epsilon \sum_{k=0}^{m} r_k \omega^k \quad , \quad 0 < \epsilon < 1$$

- r_k: independent random variables $\in [-1, 1]$

- universal scaling relation between noise strength and # terms of Padé before breakdown

$$N_c = \frac{1}{2} \frac{\log_{10}(\epsilon)}{\log_{10}(z_{\rm inf})} \quad , \quad z_{\rm inf} := \inf_{\theta \in [0, 2\pi)} \left[\psi(e^{i\theta})\right]$$