Resurgence: rigorous results

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LaTeX beamer

Resurgence bibliography, under construction, at the end.

Transseries and their universality: a bird's view

- Hardy's crucial remark: *"The only scales of infinity that are of any practical importance in analysis are those which may be constructed by means of the logarithmic and exponential functions." (Orders of Infinity, 1911 §III, 1.).* Cf. **transseries**.
- **Building transseries.** One starts with a primitive "letter", " exp " and rules of "sentence formation" by means of composition, function inversion, infinite sums with real (or complex) coefficients and products. The formal universe thus generated is the space of **transseries**.
- Example of a transseries as $x \to +\infty$:

$$\exp\left[\exp(x)\sum_{k\in\mathbb{N}} k!x^{-k-1}\right] + \sum_{l\in\mathbb{N}}(e^{-x}x^{\beta}\log x)^{k}\sum_{k\in\mathbb{N}} c_{kl}x^{-l}$$

The terms must be well ordered descendingly as $x \to +\infty$

- Independently$\sim$ 2005: Aschenbrenner, van der Hoeven, and OC proved rigorously, and : transseries are the closure of formal power series under all operations– that we could imagine. As a result, "natural" functions in analysis that have asymptotic representations at infinity should be represented by transseries. These ideas go back to Écalle.

- Let $\mathcal{L}$ be the Laplace transform. The series $\tilde{f}$ is called resurgent if its Borel transform $F = \mathcal{B}f$ ($\mathcal{B}$ = formal $\mathcal{L}^{-1}$ w.r.t. a unique power of x, the *Écalle critical time*), is
  - (1) Écalle-Borel summable. EB summation ($\mathcal{LB}$) is Borel summation enhanced by Écalle averaging to tackle singularities and Écalle acceleration to deal with possible superexponential growth. Furthermore: F has a rich set of special properties:
  - (2) endless continuability on a Riemann surface (in any bounded neighborhood of a curve on the associated universal cover, F the set of singularities is discrete)
  - (3) the singularities of F are interlinked by a set of relations, Écalle's bridge equations.
  - (4) After Écalle acceleration (rarely needed), in sectors without singularities on the first Riemann sheet, F has at most exponential growth.
- In practice, after summing all component series of a transseries, it becomes convergent, so I omit more general definitions.
- The space of EB summable transseries is **believed**, following Écalle, to be similarly closed under all operations in analysis. But so far, no one dared to engage in such a mathematically monumental, and possibly futile enterprise.
- All evidence points out that series coming from QFT, QCD etc are also resurgent. The origin of this resurgence is still shrouded in a mist.

Resurgence in generic meromorphic ODEs: "All" is known.

- Placing the **singularity at infinity** the normal form in Écalle critical time is

$$\boxed{\mathbf{y}' = (\Lambda + x^{-1}B)\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}) + \mathbf{f}_0(x^{-1}); \quad \mathbf{y} \in \mathbb{C}^n \quad (*)}$$

$$\Lambda = \operatorname{diag} \lambda_i,\ B = \operatorname{diag} \beta_i,\ \ \mathbf{f}_0 = o(x^{-m}),\ \ \mathbf{g} = O(x^{-2}, x^{-2}y, y^2)$$

Theorem (OC, Duke Math. J. 1998)

Let $\mathcal{LB}$ be the Écalle-Borel summation operator along some chosen direction.

- (i) The general formal transseries solution is

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}>1} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{\mathbf{k}\cdot\boldsymbol{\beta}} \tilde{\mathbf{y}}_{\mathbf{k}}(x), \ \ \mathbf{C} \in \mathbb{C}^n \ \ (\mathbf{C}^k := \prod_{i \leqslant n} C_i^{k_i})$$

  Well-ordering imposes the constraint $C_j = 0$ if $\operatorname{Re}(\lambda_j x) \leqslant 0$.

- (ii) The general $o(1)$ actual solution is the Écalle-Borel sum

$$\mathcal{LB}\tilde{\mathbf{y}} = \mathcal{LB}\tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}>1} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}(x),$$

- (iii) Écalle averaging is equivalent to a Laplace transform in a Banach space of generalized distributions. (iv) $\mathcal{LB}\tilde{\mathbf{y}}$ is analytic in $(x, \mathbf{C})$ for large x in the given direction.

Borel plane of $\mathbf{Y_k} = \mathcal{B}\tilde{\mathbf{y}}_\mathbf{k}$, nonlinear ODEs

- $\mathbf{Y}_0$ has equally spaced singularities at $k\lambda_j$, $k \in \mathbb{N}, j \leqslant n$.
- $\mathbf{Y}_{(3,0,\dots,0)}$ has additional singularities at $-2\lambda_1$ and $-\lambda_1$.

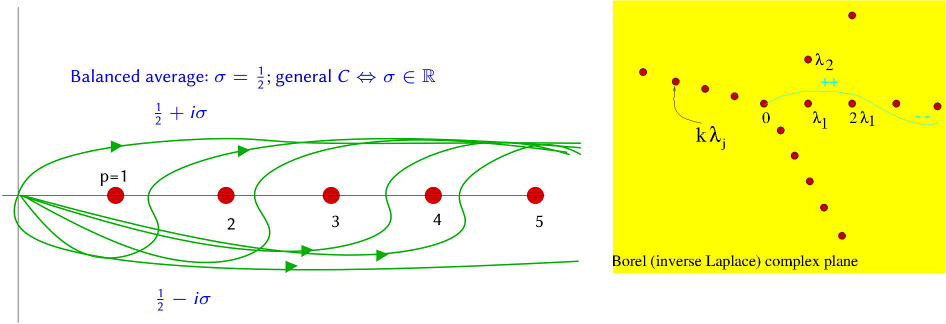

Figure: All reality-preserving averages, *in ODEs*

How resurgence relations are seen in this formalism

Theorem (OC, Duke Math. J. 1998)

i) For all $\mathbf{k}$ *and* $\mathrm{Re}\,(p) > j$, $\mathrm{Im}\,(p) > 0$

$$\mathbf{Y}_{\mathbf{k}}^{\pm j \mp}(p) - \mathbf{Y}_{\mathbf{k}}^{\pm (j-1) \mp}(p) = (\pm S_1)^j \binom{k_1 + j}{j} \left(\mathbf{Y}^{\mp}_{\mathbf{k}+j\mathbf{e}_1}(p-j) \right)^{(mj)}$$

In particular, farther singularities of $\mathbf{Y}_0$ *are obtained from analytic continuations around the first ones.*

- The paper above contains a variety of other results on the Borel plane structure, the precise shape of each singularity, bounds on the transseries, etc. See also the less general but more user-friendly OC, IMRN, 1995.
- Next, transseries breakdown and formation of singularities.

Antistokes lines. Transseries and beyond

$$y = y_0 + \sum_{k>1} C^k e^{-kx} y_k(x) = \sum_{k\geqslant 0} \xi^k y_k \quad (y_k := \mathcal{L}\mathcal{B}\tilde{y}_k)$$

- Take for simplicity $n = 1, \lambda = 1, \beta = 0$. Near antistokes lines ℓ, $x = i|x|$, the exponential becomes oscillatory, then large; the resummed transseries diverges. What happens to the solution?

- **Roughly**: At fixed x, y is a series in C, convergent in some disk D. On ∂D there **must** be singular points C_j. To leading order in x, this means singularities at the periodic points x satisfying $Ce^{-i|x|} = C_j$.

Transasymptotic matching

- Write $\tilde{y} = \sum_{k,l} c_{k,l} \xi^k x^{-l}$. Near ℓ, when $C^k e^{-kx} =: \xi \gg x^{-1}$ reexpand: Since $\xi \gg x^{-1}$, near ℓ, the dominant part of $\tilde{y}$ is $\tilde{y} \sim \sum_k c_{0k} \xi^k$.

  Next comes the x^{-1} correction: $\tilde{y} \sim \sum_k c_{0k} \xi^k + x^{-1} \sum_k c_{1k} \xi^k$. And so on.

- For a general ODE we formally get

$$\mathbf{y}(\xi_1, x) \sim \sum_{l=0}^{\infty} \mathbf{F}_l(\xi_1) x^{-l} \quad \xi_1 = C_1 e^{-\lambda_1 x} x^{\beta_1} \qquad (*)$$

Theorem (Singularities. O C, R D Costin, Invent. Math., 2001)

$(*)$ is an asymptotic expansion of $\mathbf{y}$ in the singular region near antistokes lines ℓ valid within $o(1)$ of actual singularities of $\mathbf{y}$. To leading order, $\mathbf{y}$ is singular near the singularities of F_0; the corrections are given by $\mathbf{F}_j, j \geqslant 1$.

- The singularities, are quasiperiodic since ξ_1 is. (First observed empirically.)
- For first and second order ODEs the F_k satisfy ODEs solvable by quadratures! For Painlevé equations the F_k are rational functions. In P_{I} $F_0 = \dfrac{\xi}{(\xi/12 - 1)^2}$: double poles. Integrability can be seen in Borel plane.
- A curious phenomenon: In typical nonintegrable systems, quasiperiodic small "galaxies" of branch points appear, with more singularities inside.
- Transasymptotic matching is now often used in ODEs, PDEs etc to detect singularities, or otherwise understand the new phenomena when transseries break down. Used in math to prove the Dubrovin conjecture (OC, Huang, Tanveer, Duke Math J, 2014) among many other problems.

Resurgence in PDEs

First the bad news.

- Unlike in ODEs, for which the general transseries can be generated **algorithmically**, in many **PDEs** power series/transseries **are not formally computable**.
- Take, e.g., $i\psi_t = -\Delta\psi + r^{-1}\psi + V(x)\cos\omega t$. What are the dominant terms for large t? All.
- It turns out that for large t, the coefficients of the leading power series of $\psi(x,t)$ depend on $\psi(x,0)$. I.e., the asymptotic problem is also a **connection problem**.
- Furthermore, most often, we cannot take $\mathcal{L}^{-1}$ of the PDE because analyticity in a half plane fails, as we shall see.

Now the good news.

- Note that the inverse $\mathcal{L}^{-1}$ is dual to $\mathcal{L}$ (they both are Fourier transforms!). They are equivalent. But, *at least* for linear problems, $\mathcal{L}$ is **primus inter pares** ("first among equals").

Laplace or inverse Laplace?

- Start with trivial example. $y' + y - x^{-1} = 0$; taking $\mathcal{L}(xy' + xy - 1)$ we get $-qY'(q) - Y'(q) - Y(q) - \frac{1}{q} = 0$ with solution $\frac{C}{q+1} - \frac{\log(q)}{q+1}$.
- Now $\mathcal{L}^{-1}(C/(p+1)) = Ce^{-x}$. For second term, deform the contour onto $\mathbb{R}^-$ collecting the branch jump of the log. Then take $q = -p$. We get

$$y(x) = \mathrm{PV}\int_0^\infty \frac{e^{-px}}{1-p}dp + Ce^{-x} \tag{1}$$

which is the Écalle-Borel sum **medianized** (for free!) of the general transseries solution

$$\sum_{k\geqslant 0} \frac{k!}{x^{k+1}} + Ce^{-x}$$

The expansion of y near zero gives $\log(x) + \gamma + C + o(1)$ and we also solved the connection problem!

Time independent Schrödinger

- Our research focuses on the time-periodic Schrödinger equation. To avoid becoming too technical, I'll mainly discuss the much simpler equation time-independent one:

$$i\psi_t = -\psi_{xx} + V\psi =: H\psi \tag{2}$$

with $\mathrm{supp}(V) \subset [-1, 1]$. where we assume that V and the initial condition are C^2 and compactly supported on $[-1, 1]$.

- The Laplace transform $y = \mathcal{L}\psi$ bring this problem to one of parametric resurgence,

$$y''(x) - (V(x) - iq)\, y(x) = i\psi_0(x) \tag{3}$$

- The singularities in Laplace space: a square root branch at the bottom of the continuous spectrum, and poles at all eigenvalues **and all resonances** (corresponding to Gamow a.k.a. dressed states).

Theorem

(OC, Huang) The Borel summed transseries of $\psi(x,t)$ is

$$\psi(x,t) = \mathcal{L}\mathcal{B}\tilde{\psi}(x,t) + \sum_{k=1}^{N} b_k \psi_k(x) e^{-iE_k t} + \sum_{k=1}^{\infty} g_k \Gamma_k(x) e^{-\gamma_k t} \tag{4}$$

Here $\tilde{\psi}$ is an asymptotic power series, ψ_k are eigenstates (bound states), Γ_k are Gamow states and the exponential sum is (except for degenerate cases) a lacunary Dirichlet series.

$$\gamma_k \sim const \cdot k \log k + k^2\pi^2 i/4 \text{ as } k \to +\infty \tag{5}$$

$\mathbb{R}^+$ is a **natural boundary** in t (except for very special V's).

- If ψ_0 and V have sufficient analyticity a result of OC and Huang shows that the Dirichlet series is itself resurgent and can be resummed, including for small t.
- For time-periodic potentials (next slide), in all settings we studied, the eigenstates are pushed in the complex domain and give rise to exponential decay.

PDEs: the Schrödinger equation, time-periodic

- Over the years we looked at a variety of one particle nonrelativistic models of **ionization and the photoelectric effect** with various types of potentials (including a 3d Coulomb model), modeling interaction of laser with atoms. To this day, resurgence methods are still the only ones available to get quantitative nonperturbative results.
  [arxiv, 19, 18, 18a, 18b, 04, 03, 01a, 01, 10, 10a, 09, 03a, 02, 01c, 01d, 01e, 00]
- Mathematically, we have Schrödinger's equation,

$$i\partial_t\psi(x,t) = [-\Delta + V_0(x) + V_1(x,t)]\,\psi(x,t),\ \ x \in \mathbb{R}^3, t > 0;\ \ \psi(0,x) = \psi_0$$

  - V_0 is the time-independent part of the potential. For instance, $V_0 = -C/|\mathbf{x}|$ for Hydrogen atoms.
  - $V_1(x,t)$ is the external periodic forcing; for a classical monochromatic E-M field, $V_1(x) = E \cdot x\cos(\omega t + \varphi)$.
  - Complete ionization means $\int_B |\psi|^2 dV \to 0$ as $t \to \infty$ for any bounded B.
  - We showed complete ionization in all models (though this is highly generic, it is often not easy to show) except in ones that we engineered as counterexamples.

Perturbation expansions with a limited number of terms

(OC, G. Dunne)

- In problems of high complexity in mathematics and physics often the solution can only be obtained as finitely many terms, n, of a perturbation expansion, usually divergent but resurgent. We assume after Borel transform we have **only n terms** of the Maclaurin polynomial, P_n, of F.
- The question is to calculate F from P_n as accurately as possible. What in fact is the maximal possible accuracy?
- Without further information about F, this question is (clearly) not well posed.
- But we show it becomes well posed within generic classes of functions living on some common Riemann surface Ω and with some common rate of growth.
- Ω and bounds are known apriori in the Borel plane rigorously for resurgent functions, or conjecturally in QFT, QCD, etc. They can be inferred nonrigorously but with high observed accuracy with the very methods we introduce.
- The optimal accuracy can be *dramatically* better than that obtained in established ways (such as Padé, or conformal maps).

A wealth of information in P_n is not manifest.

Some questions of interest in math and physics

- **Analytic continuation on the Riemann surface.** Understand F on higher Riemann sheets of Ω. We can often explore tens of sheets of Ω.
- **Coefficient extrapolation.** From P_9 of linear 2nd order ODEs, one can extrapolate P_{481} with relative errors in the coefficients of at most $\sim 0.1\%$.
- **Global reconstruction of a function from its asymptotic expansion in the physical domain.** Explained in the sequel.

Notations and conventions we use

- By the uniformization theorem, a sweeping generalization of the Riemann mapping theorem, any simply connected Riemann surface Ω is conformally equivalent to one of $\mathbb{D}, \mathbb{C}, \hat{\mathbb{C}}$. In all but the simplest cases, it is $\mathbb{D}$.
- Because of analyticity at zero in Borel plane Ω contains on their first Riemann sheet a disk around zero, say $\mathbb{D}$.
- We denote by $\psi : \Omega \to \mathbb{D}$ the uniformization map of Ω, and let $\varphi = \psi^{-1}$.

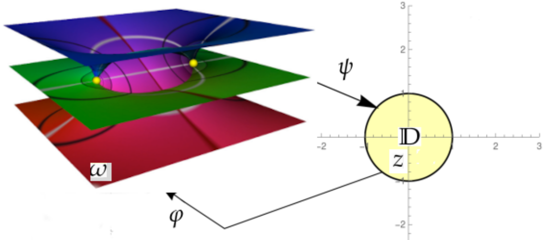

Figure: Notation. The uniformizing map $z = \psi(\omega)$ is the conformal map from the simply connected Riemann surface Ω to $\mathbb{D}$; $\varphi = \psi^{-1}$. The map is normalized as usual, $\psi(0) = 0, \psi'(0) > 0$.

The optimal reconstruction procedure

- Let $P_n(\omega)$ be the Maclaurin polynomial of an $F(\omega)$ analytic on Ω. Let ψ be the uniformization map (conformal map to $\mathbb{D}$) of Ω s.t. $\psi(0) = 0, \psi'(0) > 0$ and let $\varphi = \psi^{-1}$. Note that $\varphi(\mathbb{D}) = \Omega$ and $F \circ \varphi$ is analytic in $\mathbb{D}$.

  The most accurate reconstruction (quantified in the theorem below) of F is as follows.

  1. Take $(P_n \circ \varphi)(z)$.
  2. Expand $(P_n \circ \varphi)(z)$ in Maclaurin series at zero.
  3. **Less is more? Discard all terms** beyond the nth! We get a polynomial $(P_n \circ \varphi)_n$.
  4. The best approximant is $\boxed{\hat{R}_n := (P_n \circ \varphi)_n \circ \psi}$.
  5. To improve the accuracy of reconstruction, we needed to throw away part of the information we had (‼)

Less is more!

1. Take $(P_n \circ \varphi)(z)$. Expand $(P_n \circ \varphi)(z)$ in Maclaurin series at zero.
2. **Discard all terms** beyond the nth (!) We get a polynomial $(P_n \circ \varphi)_n$.
3. The best approximant is $\boxed{\hat{R}_n = (P_n \circ \varphi)_n \circ \psi}$.

- It seems downright **bizarre** that we need to discard information, by truncating $P_n \circ \varphi$ to $(P_n \circ \varphi)_n$ to get more accuracy... But, had we kept all terms we would've gotten precisely what we started with.
- More precisely: All terms beyond the guaranteed one, a_{n-1},

  are **exponentially wrong**. The relative error of a calculated extra term $\tilde{a}_n = (P_{n+1} - P_n) \circ \varphi$ w.r.t. a_n (unknown to us), say $0 \neq a_n$, is $1 - \tilde{a}_n / a_n = \varphi'(0)^n$. Because $\mathbb{D} \subset \Omega$ we have $\varphi'(0) > 1$. (Typically $\varphi'(0) > 2$.)

Theorem (OC, G. Dunne CMP 2022)

–Let $\omega_0 \in \Omega$, and P_n an $(n-1)$-order truncation of a Maclaurin series, converging in $\mathbb{D}$.

–Define $\mathcal{F}_n = \{F \text{ analytic on } \Omega : \|F\|_\infty < \infty, \text{ and } P_n \text{ is the Maclaurin polyn. of } F\}$

–Let $\hat{R}_n = (P_n \circ \varphi)_n \circ \psi$.

Rate of approximation: Let $\omega_0 \in \Omega$. For $F \in \mathcal{F}_n$ we have

$$\frac{|F(\omega_0) - \hat{R}_n(\omega_0)|}{\|F\|_\infty} \leqslant \frac{|\psi(\omega_0)|^n}{1 - |\psi(\omega_0)|} \ (*)$$

Optimality: $\forall R_n \in \mathbb{C}$ and $\delta > 0$ $\exists F_\delta \in \mathcal{F}_n$ s.t. $\dfrac{|F_\delta(\omega_0) - R_n|}{\|F\|_\infty} \geqslant |\psi(\omega_0)|^n(1-\delta)$

- $(*)$ Since $\Omega \supset \mathbb{D}$, we have $|\psi(\omega)| < |\omega|$ for $\omega \in \mathbb{D}$.
- **Note.** The sequence $\{(P_n \circ \varphi)_n \circ \psi\}_{n\in\mathbb{N}}$ converges on the whole of Ω.
- Weighted bounds are covered too, in the paper.
- The method is independent of ω_0 and n but is optimal at any ω_0 and any n.

Improvement of convergence: Near zero

- Say the coefficients of $P_n(\omega)$ are bounded by some c, ensuring errors $\sim c\omega^n$. Then the errors in the optimal method are $\sim \psi'(0)^n\omega^n c$; we note that $0 < \psi'(0) < 1$ is *decreasing* with the size of Ω. As an example, for $\Omega[\hat{\mathbb{C}}\backslash((0),-1,\infty)]$, the Riemann surface of classical special functions and their perturbations $\psi'(0) = 1/16$; it is rarely $> 1/2$.

Near singularities, that is near $\partial\Omega$

- Accuracy improvement is especially dramatic near singularities of F located on $\partial\Omega$. For instance, for $\Omega[\hat{\mathbb{C}} \setminus ((0),-1,\infty)]$, $\psi(-1+3\cdot 10^{-25}) = 0.9$ and $\psi(10^{12}i) = 0.9i$ meaning with n terms one can calculate these functions very far out, or very close to the singular point 1, with accuracy 0.9^n.

Riemann surface and singularities for P_{I}

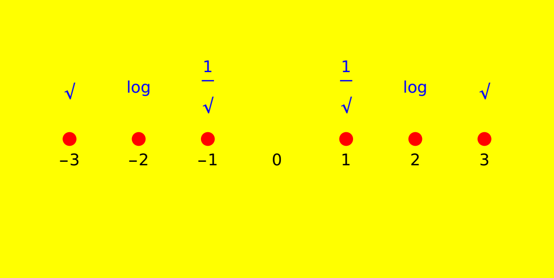

Figure: Borel plane $\Omega_{\mathbb{Z}}$ of P_{I}: universal covering of $\mathbb{C} \setminus \{(0), \mathbb{N}, -\mathbb{N}\}$.

Theorem (OC, G Dunne, CMP '22; new uniformization method)

$\Omega_{\mathbb{Z}}$ is uniformized by $\psi = \varphi^{-1}$, where $\varphi = \frac{1}{2\pi i}\ln(1 - q^{-1})$, with q the elliptic nome function.

The Riemann surface of Painlevé P_{I} mapped to $\mathbb{D}$

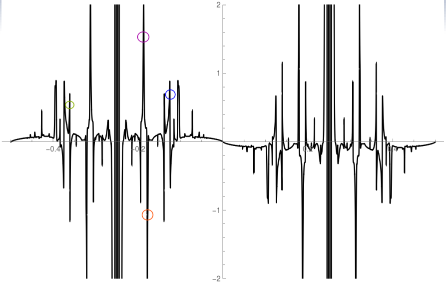

Figure: $|(P_{200} \circ \varphi)_{200}|$ of $P1$ plotted on a circle of radius 0.99, parametrized by $t \in [0, 1]$. We see singularities from many sheets.

–Notice the thick lines (the thickness decreases with the distance to $\partial\mathbb{D}$): these are two exponential singularities. The exponential nature is clear in the "large n" empirical asymptotics of the coefficients of P_{200}.

–Exponential singularities **only exist when the sheet index** $\to \infty$. We see infinitely deep on Ω, and uniformization may be the only way to extract this information from P_{200}.

Application: Global reconstruction in the physical domain

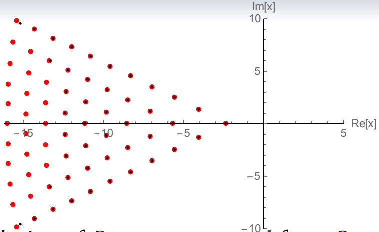

Figure: The tritronquée solution of $P1$ **reconstructed** from P_{200}, Borel transformed from its **divergent expansion as** $x \to +\infty$. There is a $2\pi/5$ wedge of poles (about 66 of which are recovered with high accuracy), and in its complement y is analytic.

- The position x_1 of the first pole and the "energy" constant h_1 at x_1 are important in applications, but not (yet?) known in closed form. Best existing numerical methods provided some 16 digits of accuracy. We get 66 digits of accuracy,

$$x_1 = -2.384168769568816639299145852448767190410408814737850512677 25\ldots$$

$$h_1 = 0.062135739226177640896490141640062460197740771373829663663 5333\ldots$$

The accuracy is roughly preserved throughout the analyticity sector.

Thank you !

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