Introduction to Holomorphic Floer Theory: brane quantization, exponential integrals and resurgence

Yan Soibelman

KANSAS STATE UNIVERSITY

RenewQuantum seminar, April 13, 2021

This is an introduction to the joint project with Maxim Kontsevich which we started in 2014. We called it Holomorphic Floer theory. Conventional Floer theory studies among other things Floer complexes of pairs of real Lagrangian submanifolds of a real smooth symplectic manifold. HFT deals with complex symplectic manifolds. Then among Lagrangian submanifolds there are those which are complex. Floer complexes between such Lagrangians enjoy specific properties which do not hold for general real Lagrangians. As a result, HFT has applications to the topics which are not obviously of "Floer-theoretical" nature. They include differential and difference equations, periodic monopoles, coisotropic branes in the "brane quantization" story, exponential integrals, resurgence of perturbative series in QFT and so on.

Plan of the talk:

a) Brief description of HFT and the generalized Riemann-Hilbert correspondence.

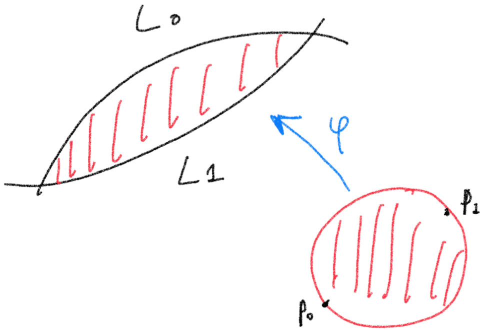
b) HFT in the special case of exponential integrals. This part is related to Maxim talks at the Renew Quantum seminar.

I will concentrate on structures and theorems rather than on formulas.

Floer complex

For a pair of transversal real Lagrangian submanifolds L_0, L_1 of a real compact symplectic manifold (M, ω) their Floer complex $CF(L_0, L_1)$ as a vector space is $Hom(L_0, L_1) = \mathbb{C}^{L_0 \cap L_1}$. This vector space is graded by the Maslov index of intersection points. The differential $d := m_1$ is defined as a sum over pseudo-holomorphic discs $\varphi : D \to M$ with two marked points p_0, p_1 at the boundary ∂D. The arc $p_0 p_1$ is mapped to L_0 and the opposite arc $p_1 p_0$ is mapped to L_1. We count each disc with the weight $e^{\int_D \varphi^*(\omega)}$. In the end we get a cochain complex over the Novikov ring $\mathbb{C}[[q^{\mathbb{R}}]] = \{\sum_{\lambda_i \to +\infty} c_i q^{\lambda_i}\}$. Precise definitions require additional assumptions and a lot of technical work. Toy model is the ordinary Morse complex of a Morse function f. It corresponds to
$M = T^*X, L_0 = X, L_1 = graph(df)$. Then $CF(L_0, L_1) = Morse(f)$. Exponential integrals in the second part of my talk are related to the same geometry, but X will be complex and f holomorphic.

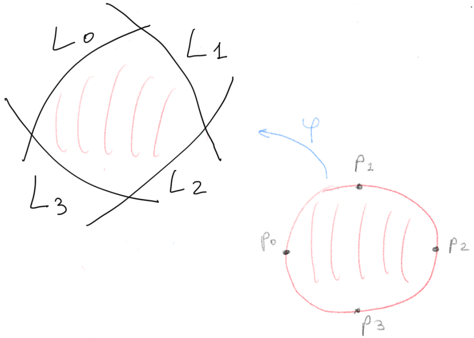
Figure for the Floer complex



Fukaya category, coisotropic branes

One can upgrade Floer complexes of pairs of Lagrangians to the A_∞-category $\mathcal{F}(M)$ (Fukaya category) with objects which are Lagrangian submanifolds endowed with local systems. For trivial rank one local systems this structure is encoded in "higher composition maps" m_n : $Hom(L_0, L_1) \otimes Hom(L_2, L_3) \otimes ... \otimes Hom(L_{n-1}, L_n) \to Hom(L_0, L_n)[2-n]$. There are versions of the notion of Fukaya category for non-compact M, there are those which include singular Lagrangians. One should also add the B-field $B \in H^2(M, U(1))$. Then instead of local systems on L one considers bundles with connections with the curvature equals $B_{|L}$. I will always assume an appropriate version of $\mathcal{F}(M)$. The construction of m_n is given in terms of moduli spaces of pseudo-holomorphic discs generalizing m_1 case. In physics they are talking about open (topological) strings ending on branes as well as about categories of A-branes. This allows physicists to discuss on the same footing A-branes with Lagrangian supports (i.e. objects of an appropriate Fukaya category) and A-branes with coisotropic support which do not have Floer-theoretical definition.

Higher compositions in Fukaya category



Generalized Riemann-Hilbert correspondence

We will consider complex symplectic manifold $(M, \omega^{2,0})$ and the category of deformation-quantization $(=DQ)$ modules on M. In order to define this category one imposes additional assumptions (as one does for $\mathcal{F}(M)$). I skip the details in this introductory talk. Morally we are talking about modules over the "quantized" sheaf $\mathcal{O}_{M,\hbar}$ of $\mathbb{C}[[\hbar]]$-algebras, which modulo $\hbar$ become sheaves of coherent modules over $\mathcal{O}_M$. If M is not compact a partial compactification is needed for controlling the support "at infinity". If the support mod $\hbar$ is a complex Lagrangian, the DQ-module is called holonomic. In general the support is only coisotropic, but not every vector bundle on a coisotropic manifold is a reduction *mod* $\hbar$ of a DQ-module. In his talk Witten spoke about $End(\mathcal{B}_{cc})$-module $Hom(\mathcal{B}_{cc}, \mathcal{B}_L)$, where $\mathcal{B}_{cc}$ is the canonical coisotropic brane supported on the whole M and $\mathcal{B}_L$ is the one supported on L. Thus $End(\mathcal{B}_{cc})$ corresponds to $\mathcal{O}_\hbar(M) = \Gamma(M, \mathcal{O}_{M,\hbar})$ and $Hom(\mathcal{B}_{cc}, \mathcal{B}_L)$ corresponds to the space of global sections of an $\mathcal{O}_{M,\hbar}$-module supported modulo $\hbar$ on L. We will be mostly interested in the case when M is a smooth complex affine algebraic variety.

The category of DQ-modules can be thought of as an enlargement of the Fukaya category by objects with coisotropic support. In order to formulate more precise conjecture, notice that we can rescale the symplectic form: $\omega^{2,0} \mapsto \omega^{2,0}/\hbar$, where $\hbar \in \mathbb{C}^*$. Assume for simplicity that M is affine algebraic, so the algebra $\mathcal{O}(M)$ of regular functions is large. Consider the family of Fukaya categories $\mathcal{F}_\hbar(M) := \mathcal{F}(M, \omega_\hbar + iB_\hbar)$ associated with the the real symplectic form $\omega_\hbar = Re(\omega^{2,0}/\hbar)$ and the B-field $B_\hbar = Im(\omega^{2,0}/\hbar)$. Notice that $\omega_\hbar$ and $B_\hbar$ vanish on $\mathbb{C}$-Lagrangians.

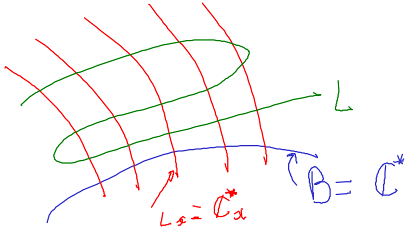
Riemann-Hilbert correspondence

In the derived sense the category of holonomic modules over the quantized algebra of holomorphic functions $\mathcal{O}_\hbar(M)$ is equivalent to the Fukaya category $\mathcal{F}_\hbar(M)$.

If the Fukaya category can be expressed in terms of constructible sheaves then the conjecture is similar to the classical RH-correspondence (Kashiwara and others). I skip here many details which are needed in order to make the conjectural RH-correspondence more precise.

Family Floer homology and RH-functor

How to construct a DQ-module corresponding to given L? HFT proposes a construction, which we illustrate on the figure below in the case when L is a spectral curve in $(\mathbb{C}^*)^2$. The family $Hom(L, L_x)$ gives in the end a module E_L over the algebra $A_q = \mathbb{C}\langle x, x^{-1}, y, y^{-1}\rangle/(xy = qyx), q = e^{\hbar}$. The case of spectral curve $L \subset T^*X$ is easier, since $Hom(L, L_x)$ is Hamiltonian invariant. Then we get a D_X-module.



$$(L, \mathrm{triv}) \longmapsto Hom\big((L, \mathrm{triv}), (L_x, \rho_x)\big)$$

Overview of the rest of the talk: exponential integrals from the point of view of HFT. It will be also an illustration of the RH-correspondence in the simplest case of D-module generated by e^f. Exponential integrals are expressions $I = \int_C e^f vol$, where f is a function on a complex manifold $X, dim_{\mathbb{C}} X = n$ endowed with a holomorphic volume form vol and C is an appropriate integration cycle, i.e. $[C] \in H_n(X, \mathbb{Z})$. From the point of view of HFT we have $M = T^*X, L_0 = X, L_1 = graph(df)$. Introducing a complex parameter $\hbar$ we can consider the exponential integral as a function $I(\hbar) = \int_C e^{f/\hbar} vol$. It can be also thought of as a "family of exponential periods" (at least when X is algebraic) hence an object of "exponential Hodge theory". In general, if L_0 and L_1 are complex Lagrangian submanifolds of $(M, \omega^{2,0})$, then for generic $\hbar$ there are no $\omega_\hbar$-pseudo-holomorphic discs with boundary on $L_0 \cup L_1$. But for "Stokes directions" $Arg(\hbar) = Arg(\int_{\gamma \in \pi_2(M, L_0 \cup L_1)} \omega^{2,0})$ such discs appear, and the Fukaya category gets changed. This is the wall-crossing phenomenon. First we will discuss exponential integrals as periods, i.e. Hodge-theoretically. Then we will explain that these results can be categorified to the statements about Fukaya categories and holonomic DQ-modules.

Data for the exponential integral

1) Smooth algebraic variety X over $\mathbb{C}$, $dim_{\mathbb{C}} X = n$.
2) Regular function $f \in \mathcal{O}(X) := \Gamma(X, \mathcal{O}_X)$.
3) Holomorphic volume form $vol := vol_X \in \Gamma(X, \Omega^n_X)$, which we assume to be nowhere vanishing (this can be relaxed).
4) Class of singular n-chains ("chains of integration").
Some of the chains (up to taking the closure of support) are of the form:
4a) $C = \sum_i k_i C_i$, where $k_i \in \mathbb{Z}_{>0}$ and each C_i is an n-chains with support which is connected real n-dimensional oriented submanifold of $X(\mathbb{C})$.
4b) We require that for each chain C the restriction $Re(f)_{|Supp(C)} : Supp(C) \to \mathbb{R}$ is a proper map bounded from above.
If we want to allow chains to have boundary, we add the following data to the list:
5a) Closed algebraic subset $D \subset X$, $dim_{\mathbb{C}} D < n$.
5b) Then we require $Supp\, \partial C \subset D(\mathbb{C})$;
Then we can define the exponential integral $I := I(f, C) = \int_C e^f vol$.

There are some other conditions on f and D which I skip here (e.g. f defines a locally trivial bundle outside of "bifurcation set"). We would like to think of the exponential integral as a pairing between the "de Rham cocycle" $e^f vol$ and "Betti cycle C". Thus we need appropriate cohomology theories with Poincaré duality.

Suppose that we can compactify X to a smooth projective variety $\overline{X}$ such that f extends to a regular map $\overline{f} : \overline{X} \to \mathbf{P}^1$. Then we have $\overline{X} - X = D_h \cup D_v$, where $D_v = \overline{f}^{-1}(\infty)$.

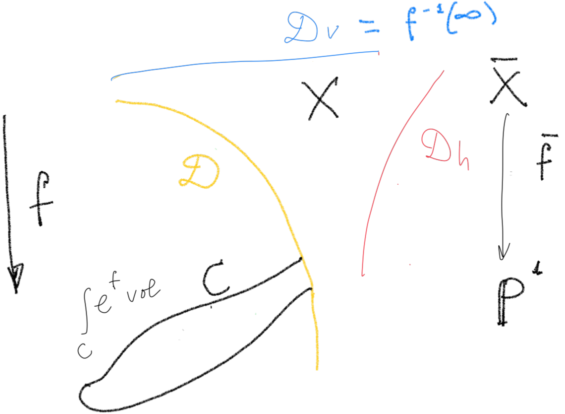
Let us assume that D, D_h, D_v are divisors with normal crossings.

Let us fix a positive real number a and consider the following singular homology of a pair:

$$H_\bullet(X, D \cup f^{-1}(Re(z) \leq -a), \mathbb{Z}) \simeq H_\bullet(X, D \cup f^{-1}(-a), \mathbb{Z}).$$

Under some conditions the homology groups stabilize as $|a|$ is sufficiently large.

Divisors



$\mathcal{D}_v = f^{-1}(\infty)$
X
$\bar{X}$
$\mathcal{D}_h$
$\bar{f}$
f
$\mathcal{D}$
$\int_C e^f vol$
C
$\mathbb{P}^1$

Betti cohomology

We define Betti homology as

$$H_\bullet^{Betti}((X,D),f,\mathbb{Z}) := H_\bullet((X,D),f^{-1}(-\infty),\mathbb{Z}),$$

where the notation in the RHS means the stabilized relative homology groups for sufficiently large $|a|$ with $-a<0$.
We define Betti cohomology in a similar way:

$$H^\bullet_{Betti}((X,D),f,\mathbb{Z}) := H^\bullet((X,D),f^{-1}(-\infty),\mathbb{Z}).$$

Betti homology and Betti cohomology are dual to each other over $\mathbb{Q}$.
Moreover, one has the following Poincaré duality:
Let $X' = \overline{X} - D_v - D$ and $D' = D_h - (D_h \cap D_v)$. Then

$$H_\bullet^{Betti}((X,D),f) \simeq H^\bullet_{Betti}((X',D'),-f)[2\,dim_{\mathbb{C}}X].$$

Twisted de Rham cohomology

Notice that under our assumptions 1)-5) the holomorphic (in fact algebraic) volume form vol_X is closed with respect to the differential $d_f = d - df \wedge (\bullet)$. The differential d_f gives rise to a complex of sheaves (in Zariski topology)

$$\Omega^\bullet_X := \Omega^0_X \to \Omega^1_X \to ... \to \Omega^n_X.$$

We denote by $\Omega^\bullet_{X,D}$ its subcomplex consisting of differential forms whose restriction to D is equal to 0.
Under the assumptions 1)-5) we define the (twisted) de Rham cohomology

$$H^\bullet_{DR}((X,D),f) = \mathbb{H}^\bullet(X_{Zar},(\Omega^\bullet_{X,D},d_f)).$$

Comparison isomorphism

The integration over cycles defines a non-degenerate pairing

$$H_{\bullet}^{Betti}((X,D),f) \otimes H^{\bullet}_{DR}((X,D),f) \to \mathbb{C}.$$

By analogy with the case $f=0$ the pairing can be thought of as the exponential period map. Hence exponential integrals can be interpreted as exponential periods of the volume form. Similarly to the case $f=0$ we have the exponential version of the comparison isomorphism of Betti and de Rham cohomology.

Theorem

We have an isomorphism of graded vector spaces
$iso : H^{\bullet}_{DR}((X,D),f) \simeq H^{\bullet}_{Betti}((X,D),f) \otimes \mathbb{C}.$

Local system of de Rham cohomology

Triples (X, D, f) generalize pairs (X, D) (i.e. $f = 0$ case). The cohomology group $H^\bullet(X, D)$ carries a mixed Hodge structure. In the case when X is projective and $D = \varnothing$ the Hodge structure is pure. The analog of that is the case $D = \varnothing$ and $f : X \to \mathbb{C}$ being a proper map. One can interpret arising Hodge-theoretical data in terms of vector bundles with connections on $\mathbb{C}$ as was proposed by Katzarkov-Kontsevich-Pantev (it is an example of their non-commutative Hodge theory). It arises such as follows. For proper f we define a family of graded $\mathbb{C}$-vector spaces $H^\bullet_{DR,\hbar}(X, f) = \mathbb{H}^\bullet(X_{Zar}(\Omega^\bullet_X, \hbar d + df \wedge (\bullet)))$. For $\hbar \in \mathbb{C}^*$ it is naturally isomorphic to the de Rham cohomology. In this way we get a vector bundle on $\mathbb{C}^*_\hbar$ endowed with a meromorphic flat connection ∇ (i.e. a D-module). It carries a covariantly constant lattice $\Gamma := (\Gamma_\hbar)_{\hbar \in \mathbb{C}^*} := iso^{-1}(H^\bullet_{Betti}(X, D), f/\hbar, \mathbb{Z})$. Assume for simplicity that all critical values $(z_i)_{i \in I}$ of f are non-zero. Then these D-modules satisfy the following properties:

nc Hodge-theoretical properties of the family of de Rham cohomology

a) For any $k \in \mathbb{Z}$ the family $H^k_{DR,\hbar}(X, f)$ gives rise to a vector bundle $E^{(k)}$ over $\mathbb{C}$.
b) The connection ∇ extends to a meromorphic connection on the vector bundle $E^{(k)}$ over $\mathbb{C}$, regular at $\hbar = \infty$. It has second order pole at $\hbar = 0$, and over the field of Laurent series $\mathbb{C}((\hbar))$ we have an isomorphism

$$(E^{(k)}, \nabla) \simeq \oplus_{i \in I} e^{z_i/\hbar} \otimes (E_i, \nabla_i),$$

where $\{z_i\}_{i \in I}$ is the set of critical values of the function f, the factor (E_i, ∇_i) is a bundle with regular singular connection, and $e^{z_i/\hbar}$ denote the irregular D-module on $\mathbb{C}$ corresponding to this exponential function.
c) Stokes filtration for $(E^{(k)}, \nabla)$ at $\hbar = 0$ is compatible with the lattice Γ.

d) Let $Z_i := f^{-1}(z_i) \cap Crit(f)$ be the "critical component" of f over the point z_i. Then over formal power series $\mathbb{C}[[\hbar]]$ we have an isomorphism

$$H^k_{DR,\hbar}(X,f) \simeq \oplus_{i\in S} H^k(U_{form}(Z_i), (\Omega^\bullet_X[[\hbar]], \hbar d + df \wedge (\bullet))),$$

where $U_{form}(Z_i)$ denotes the formal neighborhood of Z_i in X.
e) The fiber of $E^{(k)}$ at $\hbar = 0$ is isomorphic to the k-th cohomology of the complex $(\Omega^\bullet_X, df \wedge (\bullet))$ in either Zariski or analytic topology. It can be also computed as

$$\oplus_{i\in S} H^k(U(Z_i), (\Omega^\bullet_X, df \wedge (\bullet))),$$

where $U(Z_i)$ is a sufficiently small neighborhood of Z_i in analytic topology.
f) There is a non-degenerate pairing

$$H^\bullet_{DR,-\hbar}(X,f) \otimes H^\bullet_{DR,\hbar}(X,f) \to \mathbb{C}[-2\, dim_{\mathbb{C}} X],$$

which extends to a non-degenerate pairing at $\hbar = 0$.

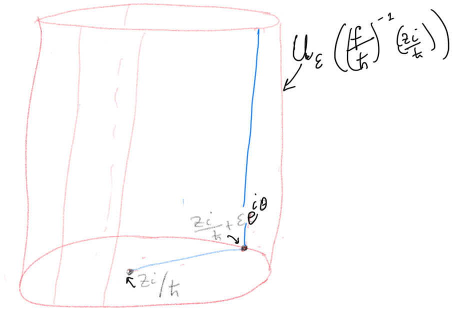
Local Betti and local de Rham cohomology

For each critical value $z_i, i \in S$ of the function f let us denote by $D(z_i, \varepsilon)$ the open disc of sufficiently small radius ε with the center at z_i. Then we have a similarly defined disc $D(z_i/\hbar, \varepsilon)$ for the function $f/\hbar$. Let us denote by $H_{Betti,loc,z_i/\hbar}$ (local Betti) an individual summand in either of the LHS or RHS of the isomorphic direct sums:

$$\oplus_{z_i \in S} H^\bullet((f/\hbar)^{-1}(z_i/\hbar), \varphi_{\frac{f-z_i}{\hbar}} \underline{\mathbb{Z}}_X[-1]) \simeq$$

$$\oplus_{z_i \in S} H^\bullet(U_\varepsilon((f/\hbar)^{-1}(z_i/\hbar)), f^{-1}(z_i/\hbar + \varepsilon \cdot e^{i\theta}, \mathbb{Z})),$$

where $\theta = Arg(\hbar)$ and $U_\varepsilon(Z)$ denote sufficiently small open neighborhood of the set Z, e.g. $U_\varepsilon((f/\hbar)^{-1}(z_i/\hbar)) = (f/\hbar)^{-1}(D(z_i/\hbar, \varepsilon))$ and φ_g is the sheaf of vanishing cycles of g.
Let $H_{DR,loc,z_i/\hbar,f/\hbar}$ (local de Rham) denote the cohomology $H^\bullet(U_\varepsilon((f/\hbar)^{-1}(z_i/\hbar)), d + df/\hbar \wedge (\bullet))$.


$\mathcal{U}_\varepsilon\left(\left(\frac{f}{\hbar}\right)^{-1}\left(\frac{z_i}{\hbar}\right)\right)$
$\frac{z_i}{\hbar}+\varepsilon e^{i\theta}$
$z_i/\hbar$

Then we have the following four isomorphisms:
1) (Betti local to global isomorphism). Outside of the Stokes rays (where ≥ 2 points z_i, z_j belong to a ray $Arg(\hbar) = const$) we have an isomorphism of local systems of abelian groups on $\mathbb{C}^*_\hbar$:

$$i_{Betti,\hbar} : H_{Betti,\hbar} \simeq \oplus_{i \in S} H_{Betti,loc,z_i/\hbar}.$$

2) (de Rham local to global isomorphism). We have the natural isomorphism of $\mathbb{C}((\hbar))$-vector spaces:

$$i_{DR,loc} : H_{DR,\hbar} \otimes \mathbb{C}((\hbar)) \simeq \oplus_{i \in S} H_{DR,loc,z_i/\hbar} \otimes \mathbb{C}((\hbar)).$$

3) (Global Betti to de Rham). For each $\hbar \in \mathbb{C}^*$ we have:

$$iso_\hbar : H_{Betti,\hbar} \otimes \mathbb{C} \simeq H_{DR,\hbar}.$$

It gives rise to an isomorphism of holomorphic vector bundles on $\mathbb{C}^*_\hbar$.
4) (Local Betti to de Rham). For each $i \in S$ we have an isomorphism of $\mathbb{C}((\hbar))$-vector spaces:

$$iso_{loc} : H_{Betti,loc,z_i/\hbar} \otimes \mathbb{C}((\hbar)) \simeq H_{DR,loc,z_i/\hbar} \otimes \mathbb{C}((\hbar)).$$

Global and local Fukaya categories

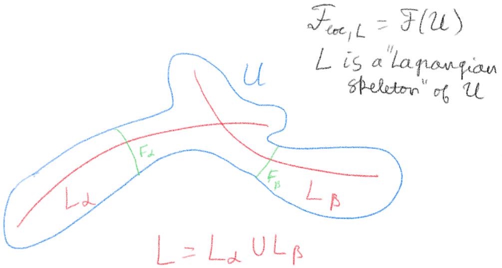
Let X be a complex smooth algebraic variety of dimension n and $M = T^*X$ denote the cotangent bundle endowed with the standard holomorphic symplectic form $\omega^{2,0}$. We split $\omega^{2,0}$ as a sum $\omega + iB$, where $\omega = Re(\omega^{2,0})$ makes M into the real symplectic manifold and $B = Im(\omega^{2,0})$ is treated as the B-field. Notice that M is *exact* symplectic manifold. Let $f \in \mathcal{O}(X)$ be a regular function. Then we have two *exact* complex Lagrangian submanifolds of the *exact* complex symplectic manifold M, namely $L_0 = X$ (zero section) and $L_1 = graph(df)$.
Let $\mathcal{F}_\hbar := \mathcal{F}_\hbar(M)$ denote the Fukaya category of M associated with the real symplectic form $\omega_\hbar = Re(\omega^{2,0}/\hbar)$ and the B-field $B_\hbar = Im(\omega^{2,0}/\hbar), \hbar \in \mathbb{C}^*$. The category $\mathcal{F}_\hbar$ depends on certain partial compactification of M, which I will not discuss here.
Then Betti cohomology can be derived from the Fukaya category. E.g.:

$$Ext^\bullet_{\mathcal{F}_\hbar}(L_0, L_1) \simeq H^\bullet(X, f^{-1}(z), \mathbb{Z})$$

as long as z belongs to the ray $\mathbb{R}_{<0} \cdot \hbar$ and $|z|$ is sufficiently large.

In general, having a Lagrangian subvariety L of a real symplectic manifold we can define a local Fukaya category $\mathcal{F}_{loc,L}$ by considering a sufficiently small Liouville neighborhood of L (conjecturally it always exists). Local Fukaya category is equivalent to the category of modules of finite rank over a $\mathbb{Z}$-graded A_∞-algebra over $\mathbb{Z}$. Let us consider $\mathcal{F}_{loc,L_0\cup L_1}$ for our L_0 and L_1. Assuming that f has only Morse critical points and working over $\mathbb{Z}$ we have $Hom(L_0, L_1) = \mathbb{Z}^{L_0\cap L_1} = \mathbb{Z}\langle x_1, ..., x_k\rangle$, where $\{x_1, ..., x_k\} = Crit(f)$. Recall that rescaling $\omega^{2,0} \mapsto \omega^{2,0}/\hbar$ introduces $\hbar$-dependence to Fukaya categories. We will see that families $\mathcal{F}_\hbar$ and $\mathcal{F}_{\hbar,loc} := \mathcal{F}_{\hbar,loc,L_0\cup L_1}$ provide categorifications of analytic vector bundles associated with global and local Betti cohomology respectively.

Fukaya category.pdf



Global and local categories of holonomic DQ-modules

By the generalized Riemann-Hilbert correspondence the category of holonomic DQ-modules $Hol_\hbar := Hol(M, \omega^{2,0}/\hbar)$ understood the sense of deformation quantization is equivalent to the Fukaya category $\mathcal{F}_\hbar = \mathcal{F}(M, Re(\omega^{2,0}/\hbar) + i\, Im(\omega^{2,0}/\hbar))$. Sometimes we can assume that $\hbar$ is a complex number, not a formal parameter. E.g. for $M = T^*X$ the category $Hol_\hbar$ is the specialization of the category of holonomic $\hbar - D$-modules on X at the complex number $\hbar$. Then to each exact Lagrangian one can assign an object of $Hol_\hbar$. In particular $\mathcal{O}_X$ corresponds to L_0 and the cyclic module generated by $e^{f/\hbar}$ corresponds to L_1. Similarly to the local Fukaya category $\mathcal{F}_{loc,L,\hbar}$, there is a local version $Hol_{loc,L,\hbar}$ of the category $Hol_\hbar$, defined as the category over $\mathbb{C}((\hbar))$. We consider local categories for $L = L_0 \cup L_1$ and skip L from the notation. We have the following categorical equivalences which imply the cohomology groups isomorphisms 1)-4):

1') (Fukaya local to global). Outside of Stokes rays in $\mathbb{C}^*_\hbar$ we have an isomorphism of analytic families of categories:

$$\mathcal{F}_\hbar \simeq \mathcal{F}_{\hbar,loc}.$$

Here Stokes rays are those rays $Arg(\hbar) = const$ for which there exist pseudo-holomorphic discs with boundaries on L_0 and L_1. One can show that they agree with the Stokes rays in 1).

2') (Holonomic local to global).

$$Hol_\hbar \otimes \mathbb{C}((\hbar)) \simeq Hol_{loc},$$

where in the LHS the notation means that corresponding category over $\mathbb{C}[\hbar]]$ with inverted $\hbar$.

3') (Global Riemann-Hilbert correspondence). We have an equivalence of analytic families of categories over $\mathbb{C}^*_\hbar$:

$$Hol_\hbar \simeq \mathcal{F}_\hbar.$$

4') (Local Riemann-Hilbert correspondence).

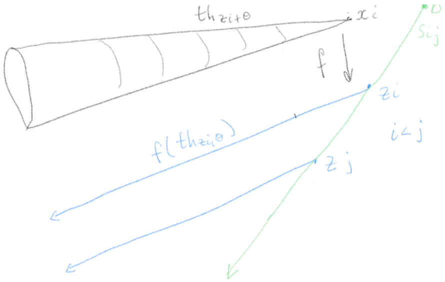
$$Hol_{loc} \simeq \mathcal{F}_{\hbar,loc} \otimes \mathbb{C}((\hbar)).$$

Thimbles

Assume now that X is Kähler and f is Morse. Let $\theta = Arg(\hbar)$. We define a *thimble* $th_{z_i,\theta+\pi}$ as the union of gradient lines (for the Kähler metric) of the function $Re(e^{-i\theta}f)$ outcoming from the critical point $x_i \in X$ such that $f(x_i) = z_i$.The same curve is an integral curve for the Hamiltonian function $Im(e^{-i\theta}f)$ with respect to the *symplectic* structure. Hence $f(th_{z_i,\theta+\pi})$ is a ray $Arg(z) = \theta + \pi$ outcoming from the critical value $z_i \in S$.
Let us assume that X carries a holomorphic volume form *vol* and define the collection of exponential integrals for all $\hbar \in \mathbb{C}^*$ which do not belong to Stokes rays $Arg(\hbar) = Arg(z_i - z_j), i \neq j$:

$$I_i(\hbar) = \int_{th_{z_i,\theta+\pi}} e^{f/\hbar} vol.$$

Assume that the set of critical values $S = \{z_1, ..., z_k\}$ is in generic position in the sense that no straight line contains three points from S. Then a Stokes ray contains two different critical values which can be ordered by their proximity to the vertex.



$th_{z_i+\theta}$
x_i
0
S_{ij}
f
z_i
$i<j$
$f(th_{z_i\theta})$
z_j

Wall-crossing formulas

It is easy to see that if in the $\hbar$-plane we cross the Stokes ray $s_{ij} := s_{\theta_{ij}}$ containing critical values $z_i, z_j, i < j$, then the integral $I_i(\hbar)$ changes such as follows:

$$I_i(\hbar) \mapsto I_i(\hbar) + n_{ij} I_j(\hbar),$$

where $n_{ij} \in \mathbb{Z}$ is the number of gradient trajectories of the function $Re(e^{i(Arg(z_i - z_j)/\hbar)} f)$ joining critical points x_i and x_j.
Let us modify the exponential integrals such as follows:

$$I_i^{mod}(\hbar) := \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{-z_i/\hbar} I_i(\hbar).$$

Then as $\hbar \to 0$ the stationary phase expansion ensures that as a formal series

$$I_i^{mod}(\hbar) = c_{i,0} + c_{i,1}\hbar + ... \in \mathbb{C}[[\hbar]],$$

where $c_{i,0} \neq 0$. The jump of the modified exponential integral across the Stokes ray s_{ij} is given by $\Delta(I_i^{mod}(\hbar)) = n_{ij} I_j^{mod}(\hbar) e^{-(z_i - z_j)/\hbar}$.

RH problem

Therefore the vector $\bar{I}^{mod}(\hbar) = (I_1^{mod}(\hbar), ..., I_k^{mod}(\hbar)), k = |S|$ satisfies the Riemann-Hilbert problem on $\mathbb{C}$ with known jumps across the Stokes rays and known asymptotic expansion as $\hbar \to 0$ (notice that because of our ordering of the points in S, the function $e^{-(z_i - z_j)/\hbar}$ has trivial Taylor expansion as $\hbar \to 0$ along the Stokes ray s_{ij}).
In abstract terms, we consider a Riemann-Hilbert problem for a sequence of $\mathbb{C}^k$-valued functions (here k is the rank of the Betti cohomology, which is under our assumptions is equal to the cardinality $|S| = k$) $\Psi_1(\hbar), ..., \Psi_k(\hbar)$ on $\mathbb{C}^* - \cup(Stokes\, rays)$ each of which has a formal power asymptotic expansion in $\mathbb{C}[[\hbar]]$ as $\hbar \to 0$, and which satisfy the following jumping conditions along the Stokes rays s_{ij}:

$$\Psi_j \mapsto \Psi_j,$$

$$\Psi_i \mapsto \Psi_i + n_{ij} e^{-\frac{z_i - z_j}{\hbar}} \Psi_j.$$

This collection $(\Psi_i)_{1 \le i \le k}$ gives rise to a holomorphic vector bundle.

The above considerations are a special case of the theory of analytic wall-crossing structures discussed in our recent paper with Maxim arXiv: 2005.10651. General resurgence conjecture formulated there "explains" the resurgence of formal expansions of exponential integrals. From the point of view of that conjecture the resurgence is equivalent to the possibility of gluing the holomorphic vector bundle over $\mathbb{C}_\hbar$ as explained above. From the point of view of the HFT the resurgence is encoded in the upper bounds for the virtual numbers of pseudo-holomorphic discs with the boundary on $L_0 \cup L_1$ which appear when $\hbar$ crosses one of the Stokes rays. Abstractly it is best of all described in the language of stability data on graded Lie algebras. In the case of resurgent series it is the Lie algebra of vector fields on the "torus of characters of the charge lattice".

On the next two slides I will review this formalism in the example of exponential integrals.

Stability data

The formalism of stability data on graded Lie algebras is convenient to studying algebraic and analytic properties of generating functions arising in Donaldson-Thomas theory, if we understand it in the sense of our original paper with Maxim arXiv:0811.2435. The data are encoded in the following:
i) free abelian group of finite rank Γ (charge lattice);
ii) graded Lie algebra $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ over $\mathbb{Q}$;
iii) homomorphism of abelian groups $Z : \Gamma \to \mathbb{C}$ (central charge);
iv) collection of elements $a(\gamma) \in \mathfrak{g}_\gamma, \gamma \in \Gamma - \{0\}$.
These data are required to satisfy one axiom called Support Property. Roughly, it says that support of the collection $(a(\gamma))_\gamma$ can be separated from *Ker Z*.

For $\mathfrak{g} := Vect_\Gamma$, the Lie algebra of vector fields on the torus $\mathbf{T}_\Gamma = Hom(\Gamma, \mathbb{C}^*)$, stability data can be restated in terms of the gluing data of a certain formal scheme. If this scheme comes from an analytic space, the stability data are called analytic. Using the $\mathbb{C}^*$ action $Z \mapsto Z/\hbar$ one can construct an analytic fiber bundle over $\mathbb{C}^*_\hbar$ with the fiber isomorphic to $\mathbf{T}_\Gamma$, roughly, by "correcting" the trivial fiber bundle by means of Stokes automorphisms which depend on the stability data. This bundle can be extended analytically to $\mathbb{C}_\hbar$ and it has a canonical trivialization at $\hbar = 0$. Then the Taylor series at $\hbar = 0$ of an analytic section of the bundle is resurgent, i.e. it is divergent, but Borel resummable. This is explained in our arXiv:2005.10651. In the case of exponential integrals with Morse function f with different critical values $z_i, 1 \leq i \leq k$, one has $\Gamma = \mathbb{Z}^k$, and $Z(e_i) = z_i, 1 \leq i \leq k$ for the standard basis $e_i, 1 \leq i \leq k$ of $\mathbb{Z}^k$. Then the gluing (Stokes) automorphisms in the standard coordinates $(x_1, ..., x_k)$ on $\mathbf{T}_\Gamma$ have the form $x_i \mapsto x_i(1 + n_{ij} e^{-Z(\gamma_{ij})/\hbar} x^{\gamma_{ij}})$, where $\gamma_{ij} = e_j - e_i, x^{\gamma_{ij}} = x_j x_i^{-1}$. Here same as before $n_{ij} \in \mathbb{Z}$ can be interpreted as the intersection index of two opposite thimbles. The Taylor expansions of $I^{mod}(\hbar)$ at $\hbar = 0$ is resurgent.

Generalizations

a) Families of functions, i.e. the function f is replaced by a family $f_u, u \in U$, e.g. $U = \mathbb{C}$, and $f(x_1, ..., x_n, u) = \frac{x_1^3}{3} - ux_1 + \sum_{i \geq 2} x_i^2$. Then we have a wall-crossing structure over $U \times \mathbb{C}^*_\hbar$.
b) Exact 1-form df is replaced by an arbitrary closed 1-form α. There is a generalization of the story with Betti and de Rham cohomology, but it is more interesting and complicated, since the form α can have non-trivial periods. From the point of view of HFT we have two complex Lagrangian submanifolds in T^*X, namely $L_0 = X, L_1 = graph(\alpha)$. Then we can take e.g. $\alpha = (\frac{1}{x} - 1)dx$ as 1-form on $\mathbb{C}^* \subset \mathbb{CP}^1$ and integrate it over the thimble $L = (0, +\infty)$ with the volume form on $\mathbb{C}^*$ given by dx/x. On L we have $\alpha = df, f = log(x) + 1 - x$. Then the corresponding version of the modified exponential integral for $\hbar > 0$ becomes $I^{mod}(\hbar) = \frac{1}{\sqrt{2\pi\hbar}} \int_L e^{\frac{1}{\hbar} f(x)} \frac{dx}{x} = \frac{\Gamma(\lambda)}{\sqrt{2\pi} e^{-\lambda} \lambda^{\lambda - 1/2}}$, where $\lambda = 1/\hbar$. This expression belongs to $\mathbb{C}[[\hbar]]$ and gives rise to a resurgent series.

Infinite-dimensional exponential integrals

Floer complex originally was defined as Morse complex on the infinite-dimensional space of paths connecting two Lagrangian manifolds. In the case when Lagrangian manifolds are complex, we can invert the logic and try to give a meaning to the non-rigorous infinite-dimensional considerations in terms of the corresponding DQ-modules. This can be useful in proving that certain perturbative series are resurgent.
It is still interesting to study the corresponding infinite-dimensional structures. Then one can speculate about the "infinite-dimensional exponential periods" or "variation of Hodge structure of infinite rank", etc. In this way one can construct the wall-crossing structures, but the proof of analyticity will be difficult. E.g. for complexified Chern-Simons one has to count the virtual number of solutions to the Kapustin-Witten equation.

Infinite-dimensional space of paths

Consider a complex symplectic manifold $(M, \omega^{2,0})$ and a pair of complex Lagrangian submanifolds L_0, L_1. Assume that $L_0 \cap L_1$ is an analytic subset of M. Let $P(L_0, L_1)$ be the set of real smooth path $\varphi : [0,1] \to M$ such that $\varphi(0) \in L_0, \varphi(1) \in L_1$. This is an infinite-dimensional smooth manifold. We denote by $\omega = Re(\omega^{2,0})$ the real symplectic form on M.The manifold $P(L_0, L_1)$ carries a closed 1-form $\eta = \int_Z \omega^{2,0}(f(t), s(t))$, where $f(t) \in P(L_0, L_1)$, and $s(t)$ is a tangent vector at the point $\varphi(t)$ (which can be identified with a small perturbation of the path $\varphi(t)$). Here Z is a 2-dimensional real cycle bounded by paths $f(t)$ and $s(t)$ and arbitrary real paths in L_0, L_1 connecting their endpoints (since $L_i, i = 0, 1$ are Lagrangian, the integral does not depend on the choice of the latter). Zeros of η are constant paths which maps of the interval $[0,1]$ to the intersection $L_0 \cap L_1$.

Path integrals and holonomic DQ-modules

In the case of a general complex symplectic manifold $(M, \omega^{2,0})$ the exponential integral with the potential (action) S can be symbolically written as

$$I = lim_{n\to\infty} \int_{\varphi \in P(L_0,L_1)} e^{S(\varphi)/\hbar}\psi_0(\varphi(s_0))\psi_1(\varphi(s_1))...\psi_n(\varphi(s_n))\mathcal{D}\varphi,$$

where $0 = s_0 < s_1 < s_2 < ... < s_n = 1$ are marked points on the interval $[0,1]$, $\psi_i(s_i)$ are "observables", and $\mathcal{D}\varphi$ is the ill-defined "Feynman measure" on the space of maps $P(L_0, L_1)$. The dictionary below gives a meaning for this ill-defined infinite-dimensional integral in terms of finite-dimensional data, associated with M, L_0, L_1. In this dictionary we interpret ψ_0 and ψ_n as "boundary conditions, i.e. elements of the line bundles $K_{L_0}^{1/2}$ and $K_{L_1}^{1/2}$ respectively, while the "bulk values" $\psi_i(\varphi(s_i)), 0 < i < n$ are interpreted as elements of the quantized algebra $\mathcal{O}_\hbar(M) = \Gamma(M, \mathcal{O}_{\hbar,M})$, where $\mathcal{O}_{\hbar,M}$ is the sheaf of quantized functions. Finally, we should encode a choice of the integration cycle.

All that amounts to the following data:
1) an element $\mu \in Ext^n_{Hol_\hbar(M)}(E_0^{DR,\hbar}, E_1^{DR,\hbar})$ corresponding to the volume form vol_X, where $n = \frac{dim_{\mathbb{C}} M}{2}$;
2) a class $\gamma \in Ext^0_{\mathcal{F}_{\hbar,loc}}(E_0^{Betti,\hbar,loc}, E_1^{Betti,\hbar,loc})$ corresponding to the integration cycle.
Then the corresponding exponential integral considered as a formal power series in $\hbar$ (i.e. the Feynman formal series expansion) is given by $I_{form}(\hbar) = \langle RH_\hbar(\mu), \gamma \rangle$, where $\langle \bullet, \bullet \rangle$ is the "Calabi-Yau pairing" between Ext^n and Ext^0 in the n-dimensional Calabi-Yau category $\mathcal{F}_{\hbar,loc}$ and $RH_\hbar$ is the Riemann-Hilbert functor. In order to get from $I_{form}(\hbar)$ the analytic function $I(\hbar)$ one should apply the Stokes automorphisms, as we did in the case of finite-dimensional exponential integrals. Therefore resurgence of the formal power series is determined by finite-dimensional data (bounds for virtual numbers of pseudo-holomorphic discs). These abstract data have very concrete meaning in examples. E.g. for $M = T^*X$ and L_0, L_1 as before with f being Morse, we have a basis in $Ext^0_{\mathcal{F}_{\hbar,loc}}(E_0^{Betti,\hbar,loc}, E_1^{Betti,\hbar,loc})$ consisting local thimbles associated with the critical points of f.

From the point of view of path integral, we are dealing with Hamiltonian $H = H(\mathbf{q}, \mathbf{p}, s) \equiv 0$, i.e.

$$S(\varphi) = \int_0^1 \sum_i p_i(s) \frac{dq_i(s)}{ds}.$$

We now define in the formal path integral over the local Lefschetz thimble to be equal to the pairing

$$\int e^{\frac{S(\varphi)}{\hbar}} \mathcal{D}\varphi = \langle \psi' | \psi \rangle,$$

where in the RHS we have the pairing of wave functions (i.e. cyclic vectors of the corresponding DQ-modules). We have a generalization of this story to the case when the action functional is

$$S(\varphi) = \int_0^1 \sum_i p_i(t) \frac{dq_i(t)}{dt} + \int_0^1 H(\mathbf{q}(t), \mathbf{p}(t), t) dt.$$

Then we give a “finite-dimensional” meaning of the LHS as $\langle \psi' | \widehat{H} | \psi \rangle$. It is done via our formalism of wave functions and quantum parallel transport. Then we can use the formalism of analytic stability data for studying resurgence of the series

$$\int_{\substack{\text{local Lefschetz thimble} \\ \text{outcoming from } \varphi_\alpha}} e^{\frac{S(\varphi)}{\hbar}} \mathcal{D}\varphi \underset{\hbar \to 0}{\sim} e^{\frac{S(\varphi_\alpha)}{\hbar}} \hbar^{-\frac{n}{2}} \cdot (c_{0,\alpha} + c_{1,\alpha}\hbar + c_{2,\alpha}\hbar^2 + \dots),$$

where $\{\varphi_\alpha\}$ are critical points of $S(\varphi)$.