Quantization by Branes and Geometric Langlands

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In this talk, I will explain work with Davide Gaiotto in which we aim to understand in terms of quantum field theory the picture developed in

P. Etinghof E. Frenkel, and D. Kazhdan, "An Analytic Version Of The Langlands Correspondence For Complex Curves," arXiv:1908.09677.

Some earlier developments: J. Teschner, "Quantization Conditions Of The Quantum Hitchin System and the Real Geometric Langlands Correspondence," arXiv:1707.07873, among others.

In the gauge theory approach (A. Kapustin and EW, arXiv:hep-th/0604151), the starting point for geometric Langlands is $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions with gauge group G, or Langlands-GNO dual group $G^\vee$. The theory based on G has a "gauge coupling constant" e and topological angle θ, which combine to a complex parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi \mathrm{i}}{e^2}.$$

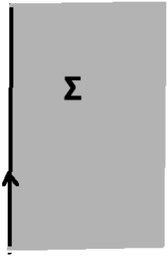
There is an elementary symmetry $\tau \cong \tau + 1$. The theory based on $G^\vee$ has an analogous parameter $\tau^\vee$, also with an elementary symmetry $\tau^\vee \cong \tau^\vee + 1$. The claim of "electric-magnetic duality," whose earliest version goes back to C. Montonen and D. Olive (1977), is that the theories based on G and on $G^\vee$ are equivalent under

$$\tau^\vee = \frac{-1}{n_{\mathfrak{g}} \tau}.$$

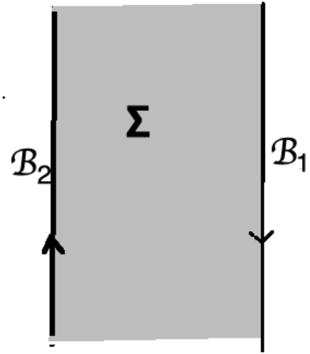
($n_{\mathfrak{g}} = 1, 2$, or 3 depending on G.)

This isn't particularly a statement about geometric Langlands, and people studying it are usually studying questions that have no obvious relation to geometric Langlands. However, we can specialize to the situation that leads to geometric Langlands. First, we consider a "topological twist" that leads to a $\mathbb{CP}^1$ family of topological field theories, parametrized say by complex parameters Ψ or $\Psi^\vee$. If we study the whole family, then the equivalence under $\Psi^\vee = -1/n_{\mathfrak{g}}\Psi$ (and $\Psi \to \Psi + 1$, $\Psi^\vee \to \Psi^\vee + 1$) becomes the duality of "quantum geometric Langlands." However, today we will consider the basic geometric Langlands duality betwen $\Psi = 0$ for G ("the A-model") and $\Psi^\vee = \infty$ for $G^\vee$ ("the B-model").

In general, quantum field theory in dimension d associates a number to a d-manifold, a vector space ("the space of physical states") to a $d-1$-manifold, and a category ("the category of boundary conditions") to a $d-2$-manifold. In the present context, since we keep the oriented two-manifold C fixed and only consider four-manifolds of the form $\Sigma \times C$ (where Σ is another two-manifold), we define a category of boundary conditions that depends on C. It is convenient to draw two-dimensional pictures in which we only exhibit Σ. If Σ has a boundary, then to formulate quantum field theory on Σ we need to specify a boundary condition

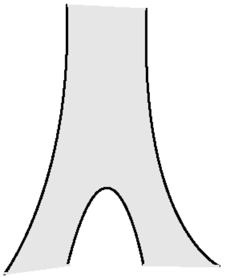


Boundary conditions make a category – given two boundary conditions $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathrm{Hom}(B_1, \mathcal{B}_2)$ is the vector space that the quantum field theory assigns to this picture



(Physicists usually don't use this terminology.)

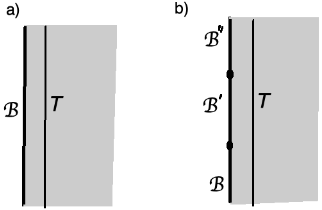
With this definition, $\mathrm{Hom}(\mathcal{B}, \mathcal{B})$ is an algebra for every $\mathcal{B}$



and moreover $\mathrm{Hom}(\mathcal{B}, \mathcal{B}')$ is a left $\mathrm{Hom}(\mathcal{B}', \mathcal{B}')$ module, etc.

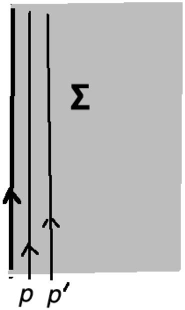
So electric-magnetic duality will give an isomorphism between the category $\mathcal{C}$ associated to C in the G theory at $\Psi = 0$ and the category $\mathcal{C}^\vee$ associated to C in the $G^\vee$ theory at $\Psi^\vee = \infty$. In addition, there is a natural mapping between certain natural functors on $\mathcal{C}$ and on $\mathcal{C}^\vee$. These functors come from what physicists call "line operators." In a quantum field theory defined on a manifold M, a line operator is some sort of modification of the definition of the theory along an embedded one-manifold $K \subset M$. The natural line operators at $\Psi = \infty$ are what physicists call "Wilson operators" (the holonomy of a connection, interpreted as an operator in quantum field theory) and the natural line operators at $\Psi = 0$ are what physicists call "'t Hooft operators." In physics, Wilson and 't Hooft operators are usually used in analyzing the confinement of quarks in atomic nuclei – and other subtleties involving the "universality classes" of quantum field theories.

But for us, a line operator is a functor from the category of boundary conditions to itself. We can understand that statement from this picture:



The point in a) is just that a line operator T that runs along a boundary with some boundary condition $\mathcal{B}$ makes a new boundary condition $T\mathcal{B}$. This explains how T acts on objects of the category. Its action on morphisms in the category is shown in b).

I've been drawing pictures in two dimensions, but there are two more dimensions not drawn and T really depends on the choice of a point $p \in C$.
Such pictures make it obvious that the line operators $T(p)$, $T(p')$, for $p, p' \in C$ commute



The picture describes $T(p')T(p)\mathcal{B}$, but for $p \neq p'$, we can move the two line operators through each other without any singularity, so $T(p)T(p')\mathcal{B} = T(p')T(p)\mathcal{B}$.

So the two dual categories $\mathcal{C}$ and $\mathcal{C}^\vee$ are equipped with dual families $T(p)$ and $W(p)$ of functors, parametrized by the choice of $p \in C$ (and some other data) and commuting at distinct points. $T(p)$ corresponds to the usual Hecke functors of geometric Langlands, while $W(p)$ corresponds to its dual in the usual geometric Langlands duality. Since these functors depend on more data (a representation R of $G^\vee$), one can also consider the composition $T_R(p)T_{R'}(p)$ of Hecke functors at the same point p but associated to possibly different representations of $G^\vee$. They commute, but a little less obviously. The duality says that the algebra of compositions $T_R(p)T_{R'}(p)$ of Hecke operators at the same point is the same as the corresponding algebra of Wilson operators $W_R(p)W_{R'}(p)$, which (on elementary grounds) is the tensor algebra of representations of $G^\vee$:

$$W_R(p)W_{R'}(p) = W_{R\otimes R'}(p),$$

where $W_{R\otimes R'}(p)$ can be expanded as a sum of irreducibles. (The fact that the decomposition of the T's matches that of the W's is known as the geometric Satake correspondence.)

To go into a little more detail, I want to introduce a useful language for formulating a simplified version of geometric Langlands duality. Let $\mathcal{M}_H(G, C)$ or just $\mathcal{M}_H(G)$ be the moduli space of G-Higgs bundles on C. As shown by Hitchin, it is a hyper-Kahler manifold. In one complex structure, I, it parametrizes Higgs bundles over C. In another complex structure, namely J, it parametrizes flat $G_{\mathbb{C}}$ bundles over C. I, J, and $K = IJ$ act as the usual unit quaterions ($IJ + JI = 0$, etc.). There are also the corresponding three Kahler forms ω_I, ω_J, and ω_K.

Geometric Langlands duality can be understood for many purposes as a mirror symmetry between the A-model of $\mathcal{M}_H(G)$ in symplectic structure ω_K and the B-model of $\mathcal{M}_H(G^\vee)$ in complex structure J. (This instance of mirror symmetry was first studied mathematically by Hausel and Thaddeus (2002).) One can definitely ask questions for which the two-dimensional picture is inadequate. Mathematicians describe this by saying that the theory should be formulated on the stack of G-bundles, Bun_G, not on a finite-dimensional moduli space. Physicists describe it by saying that the correct formulation is as a duality of four-dimensional theories. (Four is the minimum: there is a six-dimensional formulation that explains some things better, but that is not for today.) These two descriptions are not as different as you may think, since Atiyah and Bott showed in 1981 that a model of Bun_G is the space of all connections on a fixed smooth G-bundle $E \to C$; so four-dimensional gauge theory is a two-dimensional theory with target Bun_G.

I want to use the two-dimensional description of geometric Langlands duality as mirror symmetry between $\mathcal{M}_H(G)$ and $\mathcal{M}_H(G^\vee)$ to explain why the category that appears on the "automorphic" side of the duality is a category of $\mathcal{D}$-modules on Bun_G. The most familiar branes in the A-model or Fukaya category of a real symplectic manifold Y are "Lagrangian branes," supported on a Lagrangian submanifold L. However, Kapustin and Orlov discovered (2001) that in general one can also define "coisotropic A-branes" that are supported on a coisotropic submanifold $R \subset Y$ that is above the middle dimension. The construction of coistropic A-branes is delicate in general, but the simplest case is the case relevant to geometric Langlands. That is the case that Y is actually a complex symplectic manifold with complex structure I and holomorphic symplectic structure $\Omega = \omega_J + \mathrm{i}\omega_K$, where one views Y as a real symplectic manifold with symplectic form $\omega = \omega_K = \mathrm{Im}\,\Omega$, and one takes the "$B$-field" of the 2-dimensional theory to be $B = \mathrm{Re}\,\Omega = \omega_J$. (In our paper in 2006, Kapustin and I used a variant of this that differed by a "B-field gauge transformation.")

With this data one can construct a “canonical coisotropic A-brane” $\mathcal{B}_{\mathrm{cc}}$, whose support is all of Y. It has the property that, roughly, $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ is related to deformation quantization of the algebra of holomorphic functions on Y. More specifically, one can show that if $Y = T^*W$ is a cotangent bundle of some other complex manifold W with the standard complex symplectic structure, then $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ is the sheaf of holomorphic differential operators on W, acting on $K^{1/2} \to W$ (K being the canonical bundle of W).

In the case of geometric Langlands, we take Y to be $\mathcal{M}_H(G)$, the moduli space of G-Higgs bundles on C. This is birational to $T^*\mathcal{M}(G)$ where $\mathcal{M}(G)$ is the moduli space of holomorphic G-bundles on C. So $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ is the sheaf of holomorphic differential operators on $\mathcal{M}(G)$. If $\mathcal{B}$ is any other brane, then $\mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ is going to be a module for $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$. In other words, the category on the "automorphic" side of the geometric Langlands duality is a category of, roughly speaking, $\mathcal{D}$-modules on $\mathcal{M}(G)$.

Mathematically, it is important to work with $\mathcal{D}$-modules on the stack Bun_G, not on a finite-dimensional moduli space $\mathcal{M}(G)$. Physically, what that means is that it is important that a version of $\mathcal{B}_{\rm cc}$ can be defined directly in four dimensions, not only after reducing to a two-dimensional description. Kapustin and I had a version of that in our original paper, but D. Gaiotto and I found an improved version a few years later ("Knot Invariants From Four-Dimensional Gauge Theory," 2010). That also enabled us to answer a question that had been unclear and will be important later in this lecture: what is the dual of $\mathcal{B}_{\rm cc}$? $\mathcal{B}_{\rm cc}$ is an A-brane of $\mathcal{M}_H(G)$ so its dual will be a B-brane of $\mathcal{M}_H(G^\vee)$ – in other words, a coherent sheaf, or a complex of coherent sheaves, on $\mathcal{M}_H(G^\vee)$. To be more exact, what is the right sheaf was clear mathematically from the work of Beilinson and Drinfeld: the $\mathcal{D}$-module corresponding to $\mathcal{B}_{\rm cc}$ itself is the sheaf of differential operators on $\mathcal{M}(G)$, and its dual is the structure sheaf of the variety of opers, a very special Lagrangian submanifold $L_{\rm op} \subset \mathcal{M}_H(G^\vee)$. A gauge theory explanation of that fact was initially unclear, at least to me.

In summary the main ideas in the gauge theory/geometric Langlands correspondence are

- electric-magnetic duality $G \leftrightarrow G^\vee$ of supersymmetric gauge theory in four dimensions
- "twisting" to make a dual pair of topological field theories
- compactification to two dimensions on a Riemann surface C
- the dual theories have dual sets of line operators, with 't Hooft operators that are related to geometric Hecke transformations, and their dual Wilson operators
- on one side of the duality there is a distinguished brane $\mathcal{B}_{\mathrm{cc}}$, establishing a map from A-branes to $\mathcal{D}$-modules on Bun_G
- on the other side, one is studying B-branes on $\mathcal{M}_H(G^\vee)$ in the complex structure in which it parametrizes flat bundles.

This story so far involves deformation quantization, not quantization. Let me pause to underline the difference, with the concrete example of a two-sphere

$$x^2 + y^2 + z^2 = j^2$$

viewed as a symplectic manifold with its usual rotation-invariant symplectic form ($\omega = \mathrm{d}x\mathrm{d}y/z$). In deformation quantization, we start with the commutative algebra of functions $\mathbb{C}[x, y, z]$ and we want to deform it to a noncommutative algebra. In general, one specifies that the leading noncommutative deformation should agree with the Poisson bracket, and asks if a family of associative algebras over $\mathbb{C}[\hbar]$, or possibly $\mathbb{C}[[\hbar]]$, exists with that property. In the present case, we can just write down the answer, which is given by the $\mathfrak{su}(2)$ Lie algebra

$$[x, y] = -\mathrm{i}\hbar z, \quad [y, z] = -\mathrm{i}\hbar x, \quad [z, x] = -\mathrm{i}\hbar y.$$

This makes sense for any value of j^2 and is *not* quantization. Quantization means finding a Hilbert space $\mathcal{H}$ ("of the appropriate size") that the algebra acts on. This does *not* exist for arbitrary j. To construct $\mathcal{H}$, one has to "quantize" the parameter j – set it to preferred values – at which the Hilbert space exists. These special values correspond to the angular momenta in the real world of electrons, atoms, molecules, etc. It is because of this last step that the subject is called "quantization."

S. Gukov and I ("Branes and Quantization," 2008) asked whether the *A*-model in this general setup can describe quantization, and not just deformation quantization. The answer is yes, under certain conditions. Recall that in general, we are discussing a complex symplectic manifold Y (in geometric Langlands, Y is the moduli space of Higgs bundles) viewed as a real symplectic manifold with symplectic form $\omega = \mathrm{Im}\,\Omega$. Let us discuss Lagrangian *A*-branes, that is branes supported on a submanifold $L \subset Y$ that is Lagrangian for $\mathrm{Im}\,\Omega$. In many of the examples that are important in geometric Langlands, L is actually a complex Lagrangian submanifold, that is, it is Lagrangian for Ω, not just for $\mathrm{Im}\,\Omega$. (For example, the dual of a skyscraper sheaf on the $G^\vee$ side – supported on a point in $\mathcal{M}_H(G^\vee)$ – is a Lagrangian brane supported on a fiber of the Hitchin fibration of $Y = \mathcal{M}_H(G)$. This is a complex Lagrangian submanifold. That happened because a point in $\mathcal{M}_H(G^\vee)$ is "hyperholomorphic.") But in general L need not be Lagrangian for Ω (just as a coherent sheaf on $\mathcal{M}_H(G^\vee)$ need not be hyperholomorphic).

Gukov and I considered the opposite case of an A-brane $\mathcal{B}$ whose support M is Lagrangian for $\mathrm{Im}\,\Omega$ – as it must be – but is *symplectic* for $\mathrm{Re}\,\Omega$. We argued that in this case $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ represents a quantization of M with symplectic structure $\mathrm{Re}\,\Omega$. For example, if Y is an affine variety, then the holomorphic functions on Y, which are deformation quantized to get $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ can be regarded as analytic continuations of certain real analytic functions on M. There are enough of them to be a reasonable quantum-deformed algebra of observables. $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ is an $\mathcal{A}$-module which one can show has appropriate properties to be a quantization of M. A formal argument reduces the description of $\mathcal{H}$ to the problem of quantizing M.

I have actually omitted so far a key point: what structure is needed so that $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ is a Hilbert space, not just a vector space? In general, if $\mathcal{B}_1, \mathcal{B}_2$ are any two branes, there is a nondegenerate bilinear (not hermitian) pairing $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2) \otimes \mathrm{Hom}(\mathcal{B}_2, \mathcal{B}_1) \to \mathbb{C}$. To get a hermitian inner product on $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ is therefore the same as finding an antilinear map from $\mathrm{Hom}(B_2, \mathcal{B}_1)$ to $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$. Gukov and I showed that there is such an antilinear map, which I will call Θ_τ, if there is an antiholomorphic map $\tau : Y \to Y$ with M as a component of its fixed point set. I won't explain the definition of Θ_τ but I will explain why one should expect the existence of such a τ to be the right criterion. If holomorphic functions on Y act on $\mathcal{H}$ and $\mathcal{H}$ is a Hilbert space, one will have a notion of which holomorphic functions on Y are hermitian. Existence of τ gives a natural notion: a holomorphic function on Y is "real" if it is real when restricted to M.

Given τ, the hermitian product on $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ for any pair of τ-invariant branes is defined by

$$\langle \psi, \chi \rangle = (\Theta_\tau \psi, \chi)$$

where Θ_τ is the antillinear mapping that is defined using τ, and $(\ ,\ )$ is the bilinear pairing of the *A*-model. This pairing is always nondegenerate but in this generality it is not always positive definite. For the more specific case relevant to quantization, the best we can say is that the pairing on $\mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ is positive definite if one is sufficiently close to a classical limit.

In this approach to quantization, if one is given a real symplectic manifold M that one wants to quantize, one has to find a complexification of M to a complex symplectic manifold Y with some appropriate properties, and then one can use the A-model of Y to quantize M. This might be compared loosely to geometric quantization, which is the closest there has been to a systematic approach to quantization. In geometric quantization, to quantize M, one has to find a "polarization" of M (roughly a maximal set of Poisson-commuting variables). Given a polarization, geometric quantization gives a recipe of quantization. Geometric quantization – or quantization by branes – can give good results if there is a natural polarization – or a natural choice of complexification – for the problem at hand.

A limitation of my paper with Gukov is that we did not understand very much about how to compare brane quantization to geometric quantization. Gaiotto and I were motivated in the last few months to look at this more closely as background to understanding the work of Etinghof, Frenkel, and Kazhdan that I mentioned at the beginning. At some level of precision, we've been able to compare quantization by branes to geometric quantization, in the following sense: if M has a polarization $\mathcal{P}$ in the sense of geometric quantization, and a complexification Y that is suitable for quantization by branes, and if $\mathcal{P}$ analytically continues to what we call a holomorphic polarization of Y, then the two methods to quantize M agree: that is, geometric quantization of M using $\mathcal{P}$ is equivalent to brane quantization of M using Y. (For brevity, I've oversimplified what is true in the case of a complex polarization.)

All this is by way of preparation to talk about the work of Etinghof, Frenkel, and Kazhdan (EFK) that I mentioned at the start. They considered a Hilbert space $\mathcal{H}$ of L^2 functions (or better, half-densities) on $\mathcal{M}(G)$ (the moduli space of stable holomorphic G-bundles over C). They constructed operators on $\mathcal{H}$ that are related to the usual constructions of geometric Langlands, and found interesting duality theorems and conjectures concerning the action of these operators. As we proceed with the physical setup, we will see what those statements might be.

In geometric quantization, one would understand the L^2 functions on $\mathcal{M}(G)$ in terms of quantization of the cotangent bundle $T^*\mathcal{M}(G) \cong \mathcal{M}_H(G)$. In other words, the Hilbert space of EFK is what one gets if takes the Higgs bundle moduli space $\mathcal{M}_H(G)$ with real symplectic structure $\omega = \mathrm{Im}\,\Omega$, and quantizes it via geometric quantization, using the fact that it is a cotangent bundle, $\mathcal{M}_H(G) \cong T^*\mathcal{M}(G)$.

On the other hand, if we are going to get anywhere in terms of predictions from duality, we need to understand quantization of $\mathcal{M}_H(G)$ via branes. For this, the first step is to pick a complexification of $\mathcal{M}_H(G)$. Any complex manifold Y, viewed as a real manifold, has a canonical complexification, namely $\widehat{Y} = Y_1 \times Y_2$, where Y_1 and Y_2 are two copies of Y, with opposite complex structures I and $-I$. Therefore, the involution τ of $\widehat{U}$ that exchanges the two factors of Y is antiholomorphic. Its fixed point set is the diagonal $Y \subset Y_1 \times Y_2$. Since the complex structures on the two factors are opposite, we can take the holomorphic symplectic form of $\widehat{Y}$ to be $\widehat{\Omega} = \frac{1}{2}\Omega \boxplus \frac{1}{2}\overline{\Omega}$, i.e. $\frac{1}{2}\Omega$ on Y_1 and $\frac{1}{2}\overline{\Omega}$ on Y_2. Then the restriction of $\widehat{\Omega}$ to the diagonal Y is $\mathrm{Re}\,\Omega$, in other words Y is Lagrangian for $\mathrm{Im}\,\Omega$ and symplectic for $\mathrm{Re}\,\Omega$. So this is the situation in which quantization of Y, using a Lagrangian brane $\mathcal{B}$ supported on Y, makes sense. Moreover, the real polarization of Y that leads to the Hilbert space studied by EFK does analytically continue to a holomorphic polarization of $\widehat{Y}$. Hence the Hilbert space they study is the one that arises in brane quantization.

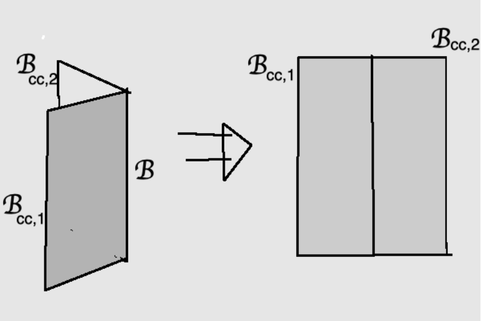
However, we have to ask what are the observables in brane quantization. Let us recall Hitchin's integrable system. The Higgs bundle moduli space has a Hitchin fibration $\pi : \mathcal{M}_H \to B$, where $B \cong \mathbb{C}^n$ for some n. The linear functions on B are Hitchin's commuting Hamiltonians. Classically the global holomorphic functions on $\mathcal{M}_H$ are just the pullbacks of functions on B, i.e. the algebra $\mathcal{A}_0$ of holomorphic functions on $\mathcal{M}_H$ is the algebra of polymomial functions of the Hitchin Hamiltonians.

What are the quantum observables in brane quantization of Y? For brane quantization, we define the canonical coisotropic brane $\widehat{\mathcal{B}}_{\rm cc}$ of $\widehat{Y} = Y_1 \times Y_2$. It is just a product $\widehat{\mathcal{B}}_{\rm cc} = \mathcal{B}_{{\rm cc},1} \times \mathcal{B}_{{\rm cc},2}$ of canonical coisotropic branes on the two factors Y_1 and Y_2. So the algebra that acts on the quantization of Y is $\mathrm{Hom}(\widehat{\mathcal{B}}_{\rm cc}, \widehat{\mathcal{B}}_{\rm cc}) = \mathcal{A} \otimes \overline{\mathcal{A}}$, where $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{{\rm cc},1}, \mathcal{B}_{{\rm cc},1})$ and $\overline{\mathcal{A}} = \mathrm{Hom}(\mathcal{B}_{{\rm cc},2}, \mathcal{B}_{{\rm cc},2})$ are quantum-deformed versions of $\mathcal{A}_0$ and $\overline{\mathcal{A}}_0$, the holomorphic and antiholomorphic functions on Y. However, for $Y = \mathcal{M}_H(G)$, one can show that the quantum-deformed rings $\mathcal{A}$ and $\overline{\mathcal{A}}$ are still commutative. This is due to Hitchin for SL_2 and to Beilinson and Drinfeld in general. The quantum deformed objects are the commuting differential operators that are the quantization of Hitchin's classical commuting Hamiltonians. The four-dimensional gauge theory picture gives another explanation of the fact that the quantum-deformed ring is still commutative, similar to the explanation I gave of the fact that Hecke functors at distinct points $p, p' \in C$ commute.

We want to understand the action of $\mathrm{Hom}(\widehat{\mathcal{B}}_{\rm cc}, \widehat{\mathcal{B}}_{\rm cc}) = \mathcal{A} \otimes \overline{\mathcal{A}}$ on $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \widehat{\mathcal{B}}_{\rm cc}) = \mathsf{L}^2(\mathcal{M}(G))$. First of all, there is no mystery about the fact that $\mathcal{A} \otimes \overline{\mathcal{A}}$ does act on $\mathsf{L}^2(\mathcal{M}(G))$. This is just the statement that holomorphic and antiholomorphic differential operators on $\mathcal{M}(G)$ can act on functions on $\mathcal{M}(G)$. Holomorphic operators trivially commute with antiholomorphic ones, and the algebras of holomorphic or antiholomorphic differential operators are separately commutative. But what are the joint eigenvalues of Hitchin's Hamiltonians and their complex conjugates? To answer this question, we want to apply duality.

To apply duality to $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \widehat{\mathcal{B}}_{\mathrm{cc}})$ where $\widehat{\mathcal{B}}_{\mathrm{cc}} = \mathcal{B}_{\mathrm{cc},1} \times \mathcal{B}_{\mathrm{cc},2}$, naively we need to understand the duals of the three branes involved, namely $\mathcal{B}_{\mathrm{cc},1}$, $\mathcal{B}_{\mathrm{cc},2}$, and $\mathcal{B}$. $\mathcal{B}_{\mathrm{cc},1}$ is the brane associated to deformation quantization of the ring of holomorphic functions, and as I explained before, its dual is the structure sheaf of the variety of opers, $L_{\mathrm{op}} \subset \mathcal{M}_H(G^\vee)$. An oper is a flat bundle whose holomorphic structure obeys a certain condition. $\mathcal{B}_{\mathrm{cc},2}$ is the brane associate to deformation quantization of the ring of antiholomorphic functions. So its dual is the structure sheaf of $L_{\overline{\mathrm{op}}}$, the Lagrangian submanifold that parametrizes flat bundles whose antiholomorphic structure obeys the oper condition. What about the brane $\mathcal{B}$ that is supported on the diagonal?

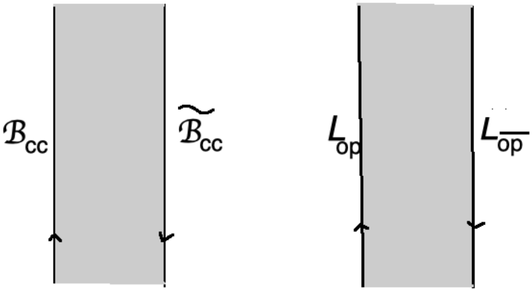
There is an "unfolding trick" that shows we do not have to worry about dualizing the diagonal:



In the unfolded version of the problem, there is just a single copy of $\mathcal{M}_H$ and the Hilbert space is $\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \widetilde{\mathcal{B}}_{\mathrm{cc}})$ where $\widetilde{\mathcal{B}}_{\mathrm{cc}}$ is a conjugate version of $\mathcal{B}_{\mathrm{cc}}$ adapted to the opposite complex structure on $\mathcal{M}_H$.

So we just dualize $\mathcal{B}_{\mathrm{cc}}$ and $\widetilde{\mathcal{B}}_{\mathrm{cc}}$ to get L_{op} and $L_{\overline{\mathrm{op}}}$, that is the varieties in $\mathcal{M}_H(G^\vee)$ that parametrize flat $G^\vee$ bundles over C whose holomorphic and antiholomorphic structures, respectively, are opers. $\mathcal{A}$ becomes $\mathrm{Hom}(L_{\mathrm{op}}, L_{\mathrm{op}})$, which is just the algebra of holomorphic functions on L_{op}, and $\overline{\mathcal{A}}$ becomes $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\overline{\mathrm{op}}})$, which is the algebra of holomorphic functions on $L_{\overline{\mathrm{op}}}$.

The two dual pictures are here:



The spectrum of $\mathcal{A} \otimes \overline{\mathcal{A}}$ is $\mathrm{Hom}(\widetilde{\mathcal{B}}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}}) = \mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$. L_{op} and $L_{\overline{\mathrm{op}}}$ are Lagrangian branes, so $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$ is an intersection or Hom space in (a version of) Floer cohomology.

I want to elaborate on this a little since the assertion that both L_{op} and $L_{\overline{\mathrm{op}}}$ are complex Lagrangian submanifolds of $\mathcal{M}_H(G^\vee)$ in the same complex structure on the latter may be confusing. The $G^\vee$ description is by a B-model of $\mathcal{M}_H(G^\vee)$, in the complex structure in which $G^\vee$ parametrizes flat $G^\vee_{\mathbb{C}}$ bundles over C. The complex structure just comes from the fact that $G^\vee_{\mathbb{C}}$ is a complex Lie group. For example, traces of holonomies are holomorphic functions on $\mathcal{M}_H(G^\vee)$ in this complex structure. A flat bundle has both a holomorphic structure and an antiholomorphic structure, both of which vary holomorphically. So in fact there is a holomorphic map $\mathcal{M}_H(G, C) \to \mathrm{Bun}_G(C) \times \mathrm{Bun}_G(\overline{C})$, where $\overline{C}$ is C with opposite complex structure. A flat bundle E (for SL_2, say) is an oper holomorphically if holomorphically it is a nontrivial extension

$$0 \to K^{1/2} \to E \to K^{-1/2} \to 0$$

and it is an oper antiholomorphically if antiholomorphically it is a nontrivial extension

$$0 \to \overline{K}^{1/2} \to E \to \overline{K}^{-1/2} \to 0.$$

By the method I explained before using an antiholomorphic involution τ, $\mathrm{Hom}(\widetilde{\mathcal{B}}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ carries a hermitian inner product, which we expect to be positive as it corresponds to the quantization of a cotangent bundle. The dual $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$ likewise carries a nondegenerate inner product which is defined in a similar way using an antiholomorphic involution $\tau^\vee$ of $\mathcal{M}_H(G^\vee)$. This nondegenerate inner product can be defined for any B-branes $L_{\overline{\mathrm{op}}}$, L_{op} that are exchanged by $\tau^\vee$. However, for generic B-branes this inner product is not positive-definite. Conditions that make the hermitian form on $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$ positive-definite were formulated by EFK, who showed that these conditions are satisfied for GL_1 (by a direct but surprisingly non-trivial computation) and for SL_2 (by a theorem of Faltings).

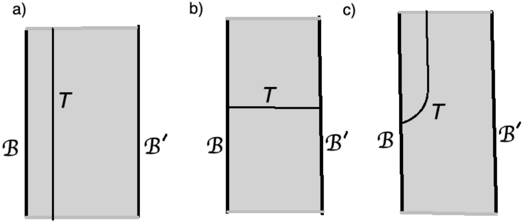
Consider a point $x \in L_{\mathrm{op}} \cap L_{\overline{\mathrm{op}}}$, corresponding to a flat bundle E that is an oper both holomorphically and antiholomorphically. The antiholomorphic involution $\tau^\vee$ that is used to define the hermitian structure on the $G^\vee$ side will map E to the complex conjugate flat bundle $\overline{E}$, which is also an oper both holomorphically and antiholomorphically. If E is not isomorphic to $\overline{E}$, then they are both null vectors for the hermitian form on $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$, which in that case is not positive-definite. But the duality predicts that it should be positive-definite, so we expect that, as conjectured by EFK and proved in some cases, E is always isomorphic to $\overline{E}$. Thus, the claim is that the flat $G_{\mathbb{C}}$ bundles that are opers both holomorphically and antiholomorphically are actually real, that is their structure group reduces to a real form of G. (I don't know if it is always the split real form.)

Assuming the intersection points are all real in that sense, $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$ is positive-definite if and only if all intersection points of $L_{\overline{\mathrm{op}}}$ and L_{op} are always transverse. This statement is also among the results/conjectures of EFK. Note that if a flat bundle E is an oper holomorphically and is also real, then it is also an oper antiholomorphically. Bundles with this property are what EFK call real opers.

Thus assuming the duality is true and the hermitian form is positive-definite, the joint spectrum of $\mathcal{A}$ and $\overline{\mathcal{A}}$ corresponds to real opers. Let me recall that according to Beilinson and Drinfeld, an element x of $\mathcal{A}$ (1) is a holomorphic differential operator on Bun_G and (2) corresponds to a holomorphic function f_x on L_{op}. Similarly an element x' of $\overline{\mathcal{A}}$ (1) is an antiholomorphic differential operator on Bun_G and (2) corresponds to a holomorphic function $f_{x'}$ on $L_{\overline{\mathrm{op}}}$. So an intersection point p of L_{op} and $L_{\overline{\mathrm{op}}}$ determines a pair of eigenvaluues of the differential operators x, x', namely $f_x(p)$ and $f_{x'}(p)$. This is the proposal for the joint spectrum of Hitchin's quantized Hamiltonians.

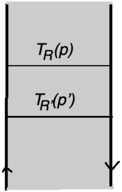
EFK also introduced Hecke operators as operators on the Hilbert space $\mathcal{H}$ that arises in quantization of $\mathcal{M}_H(G)$ as a real symplectic manifold. I explained already that line operators of the four-dimensional gauge theory can be used to define functors on the category of branes (boundary conditions). Moreover 't Hooft and Wilson line operators correspond to dual pairs of functors on the A and B model categories.

The same line operators represent operators on the quantum Hilbert space. I've tried to explain this with the following picture:



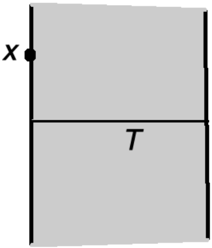
In a), I show the old story of a line operator T as a functor on the category of branes. In b), the same line operator is used to make an operator $T : \mathcal{H} = \mathcal{H}$ where $\mathcal{H} = \mathrm{Hom}(\mathcal{B}', \mathcal{B})$ (for some branes $\mathcal{B}, B'$). The picture in b) raises the question of what is happening at the corner where T ends on a brane $\mathcal{B}$. The point of c) is to explain that the corner represents an element of $\mathrm{Hom}(\mathcal{B}, T\mathcal{B})$. So actually to define the operator $T : \mathrm{Hom}(B', \mathcal{B})$ in a), one needs "junctions" $\alpha \in \mathrm{Hom}(\mathcal{B}, T\mathcal{B})$ and $\beta \in \mathrm{Hom}(T\mathcal{B}', \mathcal{B}')$.

The argument that showed commutativity of the functors $T_R(p)$, $T_{R'}(p')$ corresponding to different points $p, p' \in C$ (and representations R, R' of the dual group) goes over immediately to show that the operators $T_R(p)$, $T_{R'}(p')$ for different points $p, p' \in C$ commute. Because they live at different points in C, they can be moved up and down past each other in this picture without singularity:



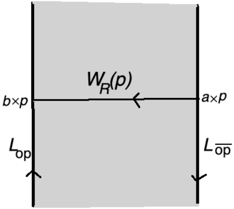
Moreover as in the discussion of line operators as functors, we can set $p = p'$. The algebra of Hecke operators $T_R(p) T_{R'}(p)$ is the same as the corresponding algebra of Hecke functors, except that when one mutliplies Hecke operators, one has to also compose the morphisms that were used to define the corners.

A similar argument shows that Hecke operators $T_R(p)$ commute with the quantized version of Hitchin's holomorphic (or antiholomorphic) Hamiltonians. A two-dimensional picture makes it look like there could be a problem in moving a Hamiltonian $x \in \mathcal{A}$ past a Hecke operator $T_R(p)$, but in four dimensions it is obvious that there is no problem:



Finally, we can ask what are the predictions of the duality for the eigenvalues of the Hecke operators. To answer this, we start on the $G^\vee$ side. We are going to find the eigenvalues of a Wilson operator $W_R(p)$ acting on $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$, and then the duality will predict that the eigenvalues of of the Hecke operator $T_R(p)$ are the same. Also, we will find what kind of data is needed for the "corners" that make $W_R(p)$ into an operator; again, the duality predicts that the same data is needed to define the Hecke operator $T_R(p)$.

Given a $G^\vee$ bundle E with connection over $\Sigma \times C$, and a representation R of $G^\vee$, we form the associated bundle $E_R = E \times_{G^\vee_{\mathbb{C}}} R$, which also comes with a connection. $W_R(\gamma)$, for a path γ in $\Sigma \times C$, is defined by parallel transport of the connection on E_R along γ. For the present application, we fix a point $p \in C$ and two points a, b on the right and left boundaries of Σ, and a path in $\Sigma \times p$ from $a \times p$ to $b \times p$:



If $E_{R,a\times p}$ and $E_{R,b\times p}$ are the fibers of E_R at $a \times p$ and $b \times p$, then parallel transport defines, for each connection, a linear transformation $W_R(p) : E_{R,a\times p} \to E_{R,b\times p}$ or equivalently an element

$$W_{R,p} :\in \mathrm{Hom}(E_{R,a\times p} \otimes E_{R',b\times p}, \mathbb{C}),$$

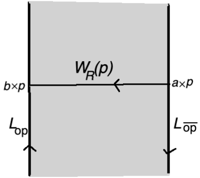
where R' is the dual representation to R.

A quantum operator is going to come from something that is a complex valued function of connnections. We do not have that yet; what we have is that for each connection we have defined

$$W_{R,p} \in \mathrm{Hom}(E_{R,a\times p} \otimes E_{R',b\times p}, \mathbb{C}).$$

To get an operator, we need to supply elements $v \in E_{R,a\times p}$, $w \in E_{R',b\times p}$ and then $W_{R,p}(v \otimes w)$ is the function of connections whose quantization will be an operator (an easy one to diagonalize and evaluate, as we will see). In order to avoid unnecessary details, I will state the following for the case that $G^\vee = SL_2$ and $R = R'$ is the two-dimensional representation, but there is a straightforward generalization to any $G^\vee$ and R.

In this picture,

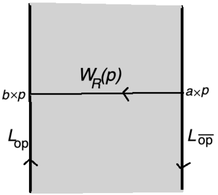


the boundary condition on the right boundary is that the $G^\vee$ bundle E, restricted to the right boundary, is an antiholomorphic oper. This means that there is a nonsplit exact sequence

$$0 \to \overline{K}^{1/2} \overset{\bar{j}}{\to} E_R \to \overline{K}^{-1/2} \to 0$$

where $\overline{K}$ is the canonical bundle of $\overline{C}$ (C with opposite complex structure) and $\overline{K}^{1/2}$ is a square root of it. (We pick here a square root, but as noted by EFK on the dual side, the choice cancels out in a moment.) So if we pick a vector $v \in \overline{K}_p^{1/2}$ (the fiber of $\overline{K}^{1/2}$ at p) then this gives a vector $\bar{j}(v) \in E_{R,a\times p}$.

Similarly on the left boundary



the boundary condition says that E is a holomorphic oper, implying that there is a nonsplit exact sequence

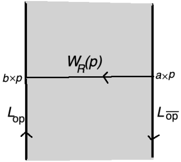
$$0 \to K^{1/2} \xrightarrow{j} E_{R'} \to K^{-1/2}.$$

So we pick $w \in K_p^{1/2}$ and this gives us $j(w) \in E_{R',b\times p}$. In other words, the "corners" correspond to $v \in \overline{K}^{1/2}$ and $w \in K^{1/2}$. Once they are picked, we define

$$\widehat{W}_{R,p} = W_{R,p}(\bar{j}(v) \otimes j(w))$$

and this is the complex-valued function of connections that we will interpret as a quantum operator.

This step is trivial, because in the B-model,



one can actually assume that the connection is pulled back from C. $\widehat{W}_{R,p}$ is just the natural dual pairing $(\ ,\ ) : E_{R,p} \otimes E_{R',p} \to \mathbb{C}$, and therefore its value at a given real oper is simply

$$(\bar{j}(v), j(w)).$$

It is convenient to write this formula in this way in terms of v, w, but it actually only depends on $v \otimes w \in K^{1/2} \otimes \overline{K}^{1/2} = |K|$, and thus in particular as observed by EFK there is no dependence on a choice of spin structure.

The duality then predicts that also on the dual side, the definition of $T(p)$ as an operator requires "corners" $v, w \in \overline{K}_p^{1/2}, \ K_p^{1/2}$. This is true, as shown by EFK using an algebro-geometric formula. The duality further predicts that the eigenvalues of the resulting operator are what we found on the $G^\vee$ side: they are

$$(\bar{j}(v), j(w))$$

for all the possible real opers.

(As a result of a recent lecture by E. Frenkel, I learned of a slight refinement. Because SL_2 has a nontrivial center $\{\pm 1\}$, parallel transport from left to right is uniquely determined only up to sign. Hence actually the eigenvalues are

$$\pm(\bar{j}(v), j(w))$$

with both signs occurring.)