From Minimal Towards Jackiw–Teitelboim Gravities: Resurgence, Resonance, Resolvent

Ricardo Schiappa

(CERN TH & University of Lisbon)

ERC ReNewQuantum SEMINAR
OnLine, February 2021

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Based on work in collaboration with:

- S. Baldino, RS, M. Schwick, R. Vega

  arXiv: Upcoming...

- P. Gregori, RS

  arXiv: Upcoming...

- I. Aniceto, RS, M. Vonk

  arXiv: Upcoming...

- B. Eynard, E. Garcia-Failde, P. Gregori, D. Lewański, A. Ooms, RS

  arXiv: Upcoming...

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...and there will be a follow-up talk in two weeks...

From Jackiw–Teitelboim Back to Minimal Gravities: Weil–Petersson, Kontsevich, Schwarzschild

Paolo Gregori

(IST — University of Lisbon)

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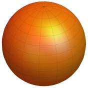
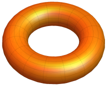
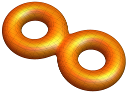
Motivation: Nonperturbative (2d) Quantum Gravity?

- String theory generically defined *perturbatively*:

$$F = \log Z \simeq \sum_{g=0}^{+\infty} F_g(t)\, g_s^{2g-2} =$$

$= \frac{1}{g_s^2}$  $+$  $+\ g_s^2$  $+ \cdots$

- Obtain nonperturbative definition/construction of string theory?
- Obtain semiclassical decoding of this nonperturbative answer? **[Mariño]**
  - Beyond perturbative $\sim g_s^{\bullet} \Rightarrow$ nonperturbative $\sim \exp\left(-\frac{\bullet}{g_s}\right)$...
- Simplest example: 2d Quantum Gravity $\equiv$ Painlevé I equation.

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2d Quantum Gravity and the Painlevé I Equation

- Painlevé I = "specific heat" of minimal strings with $c = 0$ matter:

  [Douglas-Shenker, Brézin-Kazakov, Gross-Migdal]

$$u^2 - \frac{1}{3} u'' = z.$$

- Free energy and partition function follow as:

$$F''(z) = -u(z) \quad \Rightarrow \quad Z = \exp F.$$

- String-theoretic genus expansion ($g_s = z^{-5/4}$):

$$F = \log Z \simeq \sum_{g=0}^{+\infty} g_s^{2g-2} F_g =$$

$$\simeq -\frac{4}{15} z^{\frac{5}{2}} - \frac{1}{24} \log z + \frac{7}{1440} z^{-\frac{5}{2}} + \frac{245}{41472} z^{-5} + \frac{259553}{9953280} z^{-\frac{15}{2}} + \cdots$$

- Perturbative series is asymptotic! $\Rightarrow$ Coefficients grow as $F_g \sim (2g)!$...

2d Multicritical Gravity and the Yang–Lee Equation

- Yang–Lee = "specific heat" of minimal strings with $c = -\frac{22}{5}$ matter:
  [Brézin-Marinari-Parisi]

$$u^3 - uu'' - \frac{1}{2}\left(u'\right)^2 + \frac{1}{10}u'''' = z.$$

- Free energy and partition function follow as:

$$F''(z) = -u(z) \quad \Rightarrow \quad Z = \exp F.$$

- String-theoretic genus expansion ($g_{\mathsf{s}} = z^{-7/6}$):

$$F = \log Z \simeq \sum_{g=0}^{+\infty} g_{\mathsf{s}}^{2g-2} F_g =$$

$$\simeq -\frac{9}{28} z^{\frac{7}{3}} - \frac{1}{18}\log z + \frac{1}{120} z^{-\frac{7}{3}} + \frac{247}{27216} z^{-\frac{14}{3}} + \frac{58471}{2099520} z^{-7} + \cdots.$$

- Perturbative series is asymptotic! $\Rightarrow$ Coefficients grow as $F_g \sim (2g)!\ldots$

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Minimal/Multicritical Strings from the KdV Hierarchy

- One-matrix model minimal/multicritical string series via KdV hierarchy: [Douglas-Shenker, Brézin-Kazakov, Gross-Migdal, ..., ...]

$$(2, 2k-1), \quad k = 2, 3, \ldots \quad \Rightarrow \quad c = 1 - 3\,\frac{(2k-3)^2}{2k-1}.$$

- String equation for "specific heat" = nonlinear $(2k-2)$-order ODE:

$$u^k + \sum_{i=1}^{k-1} \widehat{\alpha}_{ki}\, u^{i-1}\, u^{(2k-2i)} + \sum_{j=2}^{k-1} \widehat{\beta}_{kj}\, u^{j-2}\, u'\, u^{(2k-2j-1)} + \cdots = z.$$

- Perturbative series is asymptotic $\Rightarrow$ Coefficients grow $F_g \sim (2g)!$...
- Is $k \to +\infty$ hardest example? Not really! [Saad-Shenker-Stanford, Mertens-Turiaci]

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Jackiw–Teitelboim 2d Quantum Gravity

- Simple(st) example: 2d quantum gravity...in open/closed duality!
  [Maldacena, Sachdev-Ye-Kitaev, Almheiri-Polchinski, Maldacena-Stanford, ..., ...]

- Bulk action describes dilaton quantum gravity in AdS_2:

$$S_{\mathsf{JT}} = -\underbrace{\frac{S_0}{4\pi}\int_M \mathrm{d}^2x\,\sqrt{g}\,R}_{\text{topological}} - \underbrace{\frac{1}{2}\int_M \mathrm{d}^2x\,\sqrt{g}\,\phi\,(R+2)}_{\text{JT dilaton gravity}} + \underbrace{\int_{\partial M}\cdots}_{\text{boundary}}.$$

- Dual AdS/CFT description = quantum mechanics?

- JT gravity dual to random ensemble of quantum mechanical systems, *not* particular quantum mechanical system! [Saad-Shenker-Stanford]

- String-theoretic expansion asymptotic ⇒ ...*Resurgent*?...

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Motivation: Nonperturbative 2d Quantum Gravity

- Obtain nonperturbative definition/construction of string theory?
- Obtain semiclassical decoding of this nonperturbative answer? [Mariño]
  - Beyond perturbative $\sim g_{\mathsf{s}}^{\bullet} \Rightarrow$ nonperturbative $\sim \exp\left(-\frac{\bullet}{g_{\mathsf{s}}}\right)$...
- **RE**SURGENT transseries construction ...... requires Stokes data.
  - Nonlinear Stokes data = infinite set of (transcendental) “numbers”...
- Resurgent transseries construction is **RE**SONANT.
- **RE**SOLVENT underlies the resurgent transseries construction.

Outline

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The Painlevé I Equation Reloaded: Perturbative

- Reconsider Painlevé I equation:

$$u^2 - \frac{1}{3} u'' = z.$$

- Its perturbative solution (at large z)

$$u(z) \simeq \sqrt{z} \sum_{g=0}^{+\infty} \frac{u_g}{z^{\frac{5}{2}g}},$$

yields recursion equation; leading to asymptotic expansion:

$$u \simeq \sqrt{z} \left(1 - \frac{1}{24} z^{-\frac{5}{2}} - \frac{49}{1152} z^{-5} - \frac{1225}{6912} z^{-\frac{15}{2}} - \frac{4412401}{2654208} z^{-10} - \cdots \right).$$

- Second order differential equation ⇒ Yields *two* instanton actions:

[Garoufalidis-Its-Kapaev-Mariño]

$$A = \pm \frac{4}{5} \sqrt{6}.$$

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Two-Parameter Transseries Solution: Nonperturbative

- General two-parameter transseries solution (resonant...):

$$u(g_{\mathsf{s}},\sigma_1,\sigma_2)=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\sigma_1^n\sigma_2^m\,\mathrm{e}^{-(n-m)\frac{A}{g_{\mathsf{s}}}}\left(\sum_{k=0}^{\min(n,m)}\log^k(g_{\mathsf{s}})\cdot\Phi_{(n|m)}^{[k]}(g_{\mathsf{s}})\right).$$

  Trans-monomials ordering: $\mathrm{e}^{-\frac{A}{g_{\mathsf{s}}}}\ll g_{\mathsf{s}}\ll\log(g_{\mathsf{s}})\ll\mathrm{e}^{+\frac{A}{g_{\mathsf{s}}}}$.

- Asymptotic sectors have different starting orders β_{nm}, **[Aniceto-RS-Vonk]**

$$\Phi_{(n|m)}\simeq\sum_{g=0}^{+\infty}u_g^{(n|m)}\,g_{\mathsf{s}}^{g+\beta_{nm}}.$$

  Log-sectors not independent: $\Phi_{(n|m)}^{[k]}=\frac{1}{k!}\left(\frac{8(m-n)}{\sqrt{6}}\right)^k\Phi_{(n-k|m-k)}^{[0]}$.

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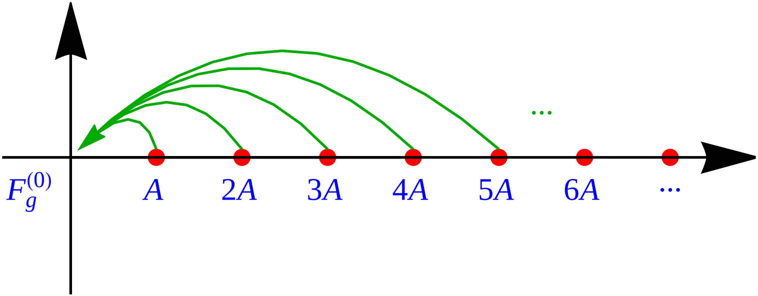
Asymptotics of (One-Parameter) Perturbative Series



- From perturbative expansion and Cauchy dispersion relation,

$$F_g^{(0)} \simeq \frac{S_1}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta)}{A^{g-\beta}} \left(F_1^{(1)} + \frac{A}{g-\beta-1} F_2^{(1)} + \cdots \right) + \\ + \frac{S_1^2}{2\pi\mathrm{i}} \frac{\Gamma(g-2\beta)}{(2A)^{g-2\beta}} \left(F_1^{(2)} + \frac{2A}{g-2\beta-1} F_2^{(2)} + \cdots \right) + \cdots$$

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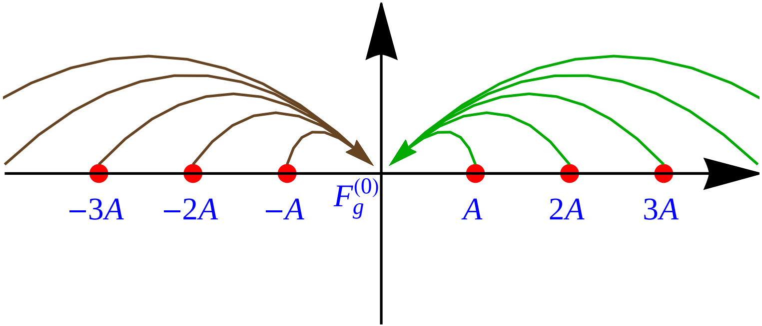
Asymptotics of *Resonant* Perturbative Series



- Two-parameter transseries with instanton actions A and $-A$...
- Already at perturbative level leading asymptotics clearly distinct:

[Aniceto-RS-Vonk]

$$F_g^{(0|0)} \simeq \frac{S_1^{(0)}}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta)}{A^{g-\beta}} F_1^{(1|0)} + \frac{\widetilde{S}_{-1}^{(0)}}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta)}{(-A)^{g-\beta}} F_1^{(0|1)} + \cdots$$

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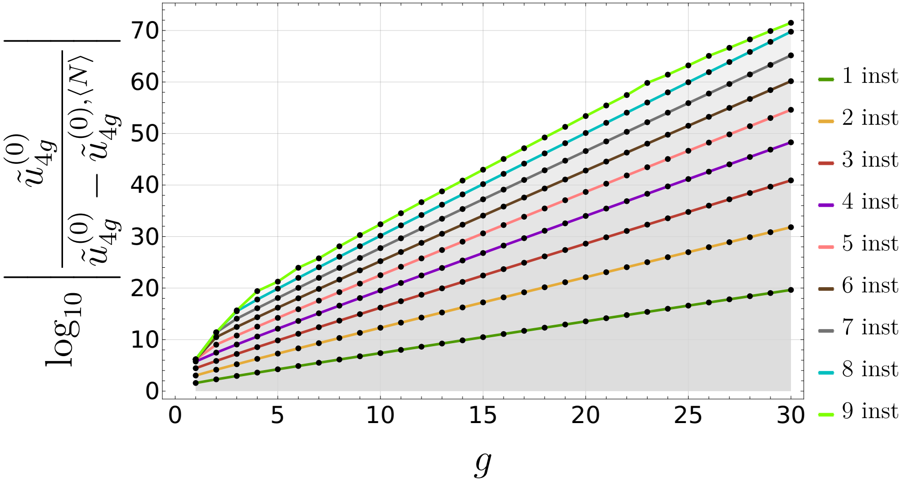
Asymptotic Checks of Resurgence: Painlevé I



Two-Parameter Transseries Resummations?

- Transseries resummation in two steps:
  1. Borel resummation of each transseries "building block",

$$\mathcal{S}_\theta \Phi_{(n|m)}(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{d}s\, \mathcal{B}[\Phi_{(n|m)}](s)\, \mathrm{e}^{-zs}.$$

  2. Écalle summation of all transseries "building blocks",

$$\mathcal{S}_\theta u = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m\, \mathrm{e}^{-(n-m)\frac{A}{g_s}} \left(\sum_{k=0}^{\min(n,m)} \log^k(g_s) \cdot \mathcal{S}_\theta \Phi_{(n|m)}^{[k]} \right).$$

  3. Holds in Stokes wedges: still *need* Stokes data...
- Further, two-parameter transseries solution are resonant:

$$u(g_s, \sigma_1, \sigma_2) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m\, \mathrm{e}^{-(n-m)\frac{A}{g_s}} \left(\sum_{k=0}^{\min(n,m)} \log^k(g_s) \cdot \Phi_{(n|m)}^{[k]}(g_s) \right).$$

- When $n = m \Rightarrow$ all diagonal *nonperturbative* sectors of same weight!

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Multi-Parameter Transseries and Borel Singularities

- k distinct instanton actions, $\boldsymbol{A} \equiv (A_1, \ldots, A_k)$, with transseries:

$$u\left(g_{\mathsf{s}}, \boldsymbol{\sigma}\right) = \sum_{\boldsymbol{n} \in \mathbb{N}_0^k} \boldsymbol{\sigma}^{\boldsymbol{n}}\, \mathrm{e}^{-\frac{\boldsymbol{n}\cdot\boldsymbol{A}}{g_{\mathsf{s}}}} \underbrace{\Phi_{\boldsymbol{n}}(g_{\mathsf{s}})}_{\simeq \sum_{g=0}^{+\infty} u_g^{(\boldsymbol{n})} g_{\mathsf{s}}^g} .$$

- Location of Borel singularities via projection map from $\boldsymbol{A}$,

$$\begin{aligned} \mathfrak{P} : \mathbb{Z}^k &\to \mathbb{C} \\ \boldsymbol{\ell} &\mapsto \boldsymbol{A}\cdot\boldsymbol{\ell} \end{aligned}$$

- Simple resurgent functions, singularities = pole + *log*-branch-cut:

$$\mathcal{B}[\Phi_{\boldsymbol{n}}](s)\Big|_{s=\boldsymbol{\ell}\cdot\boldsymbol{A}} = \underbrace{\mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}+\boldsymbol{\ell}}}_{\text{Borel residues}} \times \mathcal{B}[\Phi_{\boldsymbol{n}+\boldsymbol{\ell}}](s-\boldsymbol{\ell}\cdot\boldsymbol{A})\, \frac{\log\left(s-\boldsymbol{\ell}\cdot\boldsymbol{A}\right)}{2\pi\mathrm{i}} .$$

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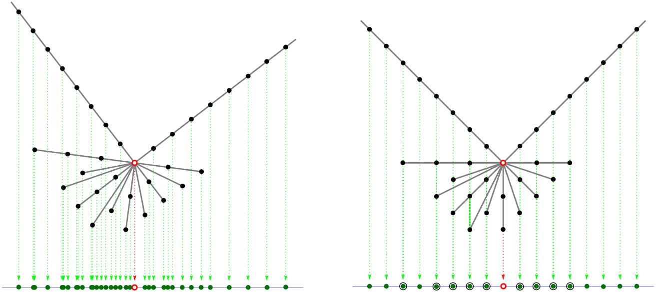
Resonant Asymptotics of (Two-Parameter) Transseries

- From transseries lattice to Borel plane: $\mathfrak{P} : \boldsymbol{\ell} \in \mathbb{Z}^k \mapsto \boldsymbol{A} \cdot \boldsymbol{\ell} \in \mathbb{C}$
- Projection map: non-resonant $\ker \mathfrak{P} = \mathbf{0}$ *vs.* resonant $\ker \mathfrak{P} \neq \mathbf{0}$.



- String theoretic examples always resonant: open *and* closed strings ✓

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Multi-Parameter Transseries and Alien Derivatives

- Alien derivatives computed from bridge equation: [Écalle]

$$\Delta_{\boldsymbol{\ell}\cdot\boldsymbol{A}}\Phi_{\boldsymbol{n}} = \boldsymbol{S}_{\boldsymbol{\ell}}\cdot(\boldsymbol{n}+\boldsymbol{\ell})\,\Phi_{\boldsymbol{n}+\boldsymbol{\ell}}$$

$\Rightarrow$ Leads to k-dimensional *vector* of Stokes coefficients (infinite set of transcendental numbers = hard!):

$$\boldsymbol{S}_{\boldsymbol{\ell}} \equiv \left(S_{\boldsymbol{\ell}}^{(1)},\ldots,S_{\boldsymbol{\ell}}^{(k)}\right).$$

- Not all lattice sites $\boldsymbol{\ell}\in\mathbb{Z}^k$ have Stokes vector,

$$S_{\boldsymbol{\ell}}^{(j)} = 0 \quad \text{if} \quad \ell_i \geq 1+\delta_{ij}, \quad \forall_{i\in\{1,\ldots,k\}}.$$

- Stokes vectors and Borel residues relate to each other, *e.g.*,

$$\begin{aligned} \mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}-2\boldsymbol{v}} &= -(\boldsymbol{n}-2\boldsymbol{v})\cdot\left\{\boldsymbol{S}_{-2\boldsymbol{v}}+\frac{1}{2}(\boldsymbol{n}-\boldsymbol{v})\cdot\boldsymbol{S}_{-\boldsymbol{v}}\;\boldsymbol{S}_{-\boldsymbol{v}}\right\}, \\ (\boldsymbol{n}-2\boldsymbol{v})\cdot\boldsymbol{S}_{-2\boldsymbol{v}} &= -\mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}-2\boldsymbol{v}}-\frac{1}{2}\,\mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}-\boldsymbol{v}}\,\mathsf{S}_{\boldsymbol{n}-\boldsymbol{v}\to\boldsymbol{n}-2\boldsymbol{v}}. \end{aligned}$$

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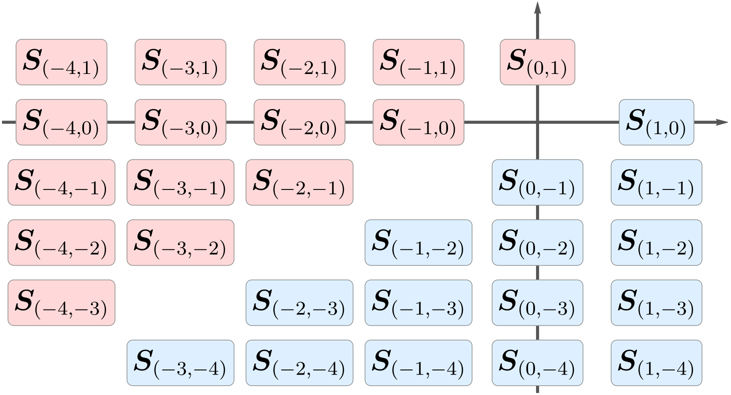
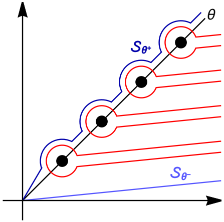
Vectorial Stokes Data for “Resurgence Lattice”



Stokes Phenomena upon Crossing a Stokes Line



- Pick $\mathbb{Z}^k$ canonical-basis versor $\boldsymbol{e}_i$, along forward direction $\Rightarrow$ projects to direction angle $\theta_i \equiv \arg A_i$ on Borel plane...
- Stokes vector only has one component, $\boldsymbol{S}_{\boldsymbol{e}_i} = S^{(i)}_{\boldsymbol{e}_i}\, \boldsymbol{e}_i$...
- Forward Stokes automorphism

$$\underline{\mathfrak{S}}_{\theta_{\boldsymbol{\ell}}} \Phi_{\boldsymbol{n}} = \exp\left(\mathrm{e}^{-\frac{\boldsymbol{\ell}\cdot\boldsymbol{A}}{x}}\, \Delta_{\boldsymbol{\ell}\cdot\boldsymbol{A}}\right) \Phi_{\boldsymbol{n}}$$

on transseries is Stokes phenomena, $\underline{\mathfrak{S}}_{\theta_i} u\left(g_{\mathsf{s}}, \boldsymbol{\sigma}\right) = u\left(g_{\mathsf{s}}, \boldsymbol{\sigma} + \boldsymbol{S}_{\boldsymbol{e}_i}\right)$.

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Organization of Resonant Stokes Data

- Transseries is resonant: $\boldsymbol{A} = (A, -A)$ and $\ker \mathfrak{P} = \mathcal{L}\{(1,1)\}$.
- Resonance yields specific structure of alien derivatives ($s \in \mathbb{N}^+$):

$$\Delta_{sA} \Phi_{(n,m)} = \sum_{p=s-1}^{\min(n+s,m)} \boldsymbol{S}_{(s-p,-p)} \cdot (n+s-p, m-p)\, \Phi_{(n+s-p,m-p)},$$

$$\Delta_{-sA} \Phi_{(n,m)} = \sum_{p=s-1}^{\min(n,m+s)} \boldsymbol{S}_{(-p,s-p)} \cdot (n-p, m+s-p)\, \Phi_{(n-p,m+s-p)}.$$

- Real positive and real negative directions now "symmetric" ✓
- Δ_{sA} sends $\Phi_{(n,m)}$ in linear combination of $\Phi_{(n+s-p,m-p)}$ with $p \leq m$ $\Rightarrow$ Organize along directions of $\ker \mathfrak{P}$.

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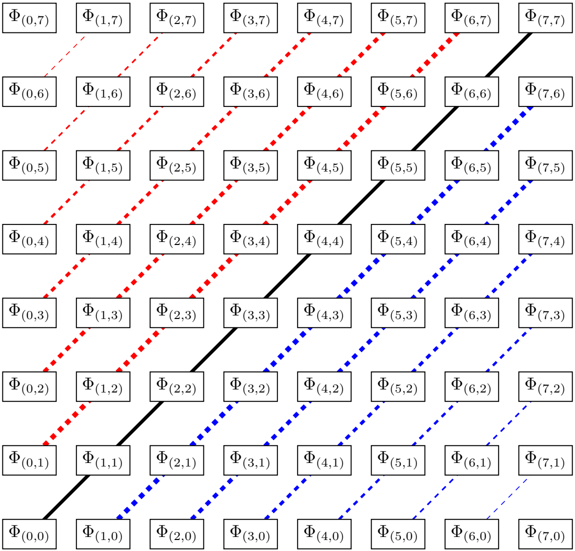
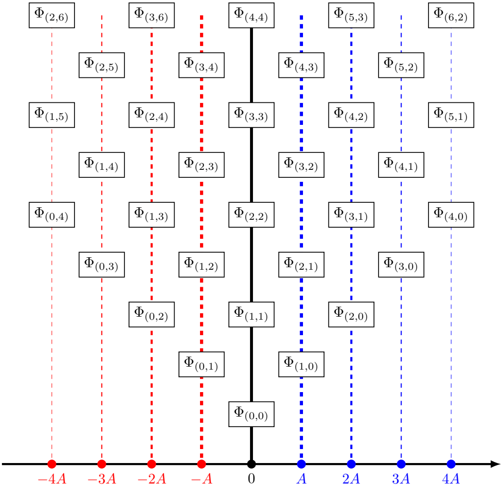
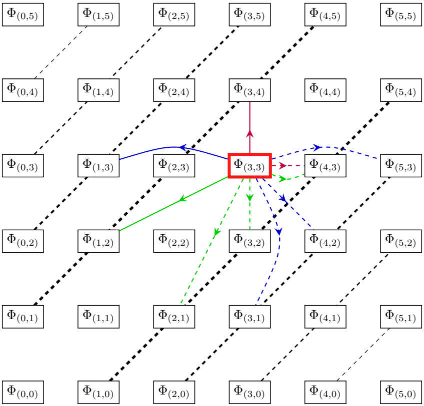
Organization of Resonant Transseries Sectors



Organization of Resonant Borel Plane

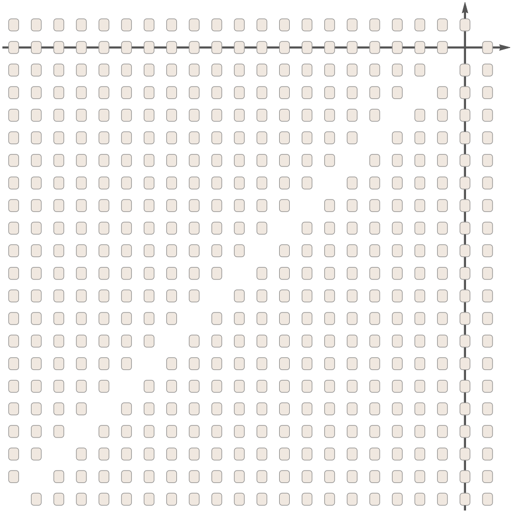


Resonance and the Backward-Forward Relation

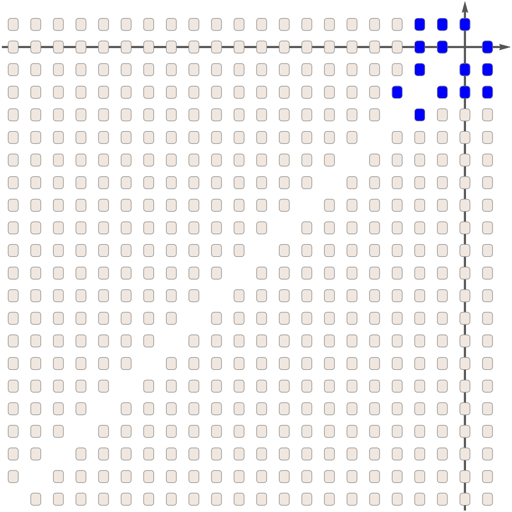


- Relates Borel residues from symmetrical diagonals (to $\ker \mathfrak{P}$) $\Rightarrow$ "Forward" Stokes $\left\{ \boldsymbol{S}_{(s-p,-p)} \right\}$ yields "backward" Stokes $\left\{ \boldsymbol{S}_{(-q,t-q)} \right\}$
- Lines of same color $=$ on same backward-forward relation.

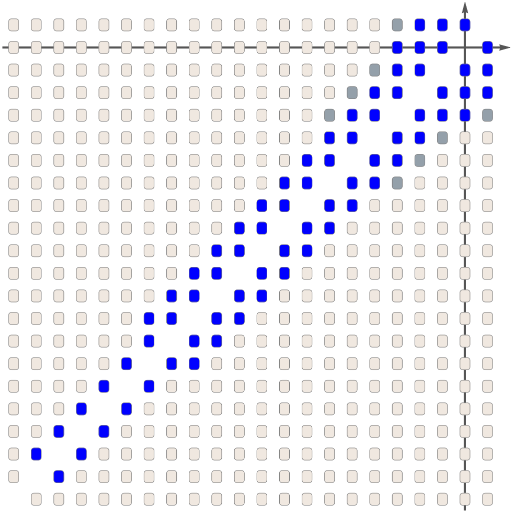
Computation of Resonant Stokes Data?



Resonant Stokes Data: *Numerical* Asymptotics $\sim$ 2011



Resonant Stokes Data: *Numerical* Residues $\sim$ 2019



Computational Precision *versus* Computational Time

- Sources of errors: Borel–Padé resummations, higher-action contributions, reconstruction of Stokes data...
- Down same diagonal: precision drops, timing increases...
- Down higher diagonals: severe precision drops, timing increases...

Precision			
$N_x^{(s)}$	$s=1$	$s=2$	$s=3$
$x=1$	~ 100	~ 50	~ 20
$x=0$	~ 100	~ 50	~ 20
$x=-1$	~ 100	~ 50	~ 10
$x=-2$	~ 70	~ 40	
$x=-3$	~ 70	~ 40	
$x=-4$	~ 70	~ 40	
$\vdots$			

Timing (2.8GHz)			
$N_x^{(s)}$	$s=1$	$s=2$	$s=3$
$x=1$	~ 3 h	~ 8 h	~ 20 h
$x=0$	~ 8 h	~ 1 d	~ 2 d
$x=-1$	~ 1 d	~ 2 d	~ 4 d
$x=-2$	~ 2 d	~ 3 d	
$x=-3$	~ 3 d	~ 5 d	
$x=-4$	~ 5 d	~ 7 d	
$\vdots$			

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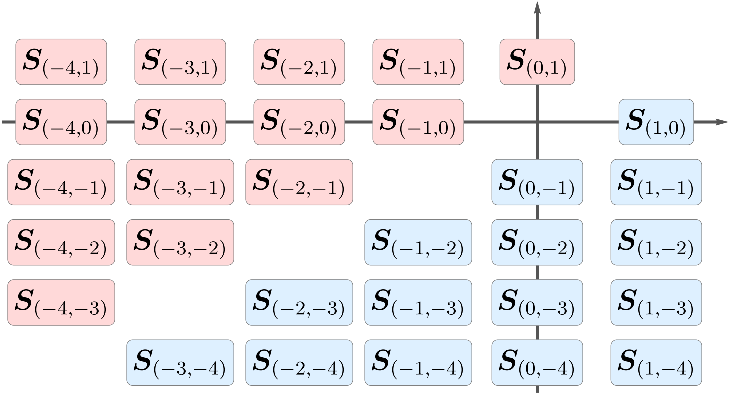
Handling the Data: *Analytical* Stokes Data?



Structure of Stokes Vectors and Alien Algebra

- Find closure of alien-lattice algebraic structure:

$$[\Delta_{\boldsymbol{n}\cdot\boldsymbol{A}}, \Delta_{\boldsymbol{m}\cdot\boldsymbol{A}}] \propto \Delta_{(\boldsymbol{n}+\boldsymbol{m})\cdot\boldsymbol{A}},$$

as, up to actions A and $2A$, following relation holds numerically:

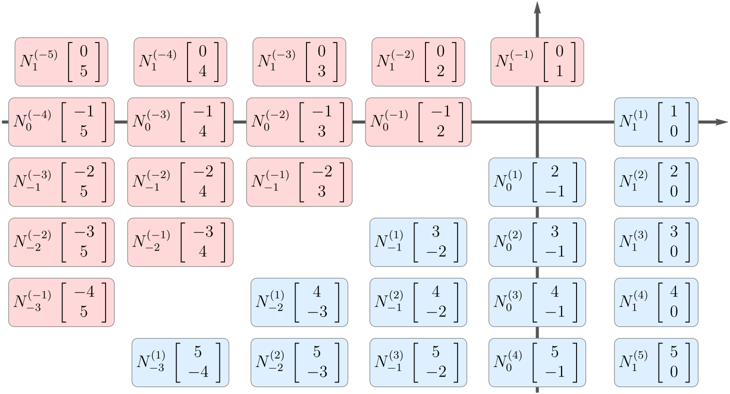
$$(\boldsymbol{S_n}\cdot\boldsymbol{m})\,\boldsymbol{S_m} - (\boldsymbol{S_m}\cdot\boldsymbol{n})\,\boldsymbol{S_n} \propto \boldsymbol{S_{n+m}} \quad \checkmark$$

- If true for any action, implies:

$$\boldsymbol{S}_{(s-r,-r)} = N^{(s)}_{s-r}\begin{bmatrix} r+1 \\ s-r-1 \end{bmatrix}, \qquad \boldsymbol{S}_{(-r,s-r)} = N^{(-s)}_{s-r}\begin{bmatrix} s-r-1 \\ r+1 \end{bmatrix}.$$

  - $\Rightarrow$ all Stokes data follows from proportionality constants alone...
  - $\Rightarrow$ all $N^{(s)}_{x}$ reconstructed from $\mathsf{S}_{(n,n)\to(s,0)}$ alone...

Data Conjectures: *Analytical* Stokes Data



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Predicting Analytical Stokes Data from Numerics?

- "Walking down" *first* diagonal, numerics yield analytics?

$$N_{-1}^{(1)} = \frac{1}{2!}\left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^1 N_0^{(1)} - \frac{1}{0!}\left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^0 \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(1)}\zeta(2),$$

$$N_{-2}^{(1)} = \frac{1}{3!}\left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^2 N_0^{(1)} - \frac{1}{1!}\left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^1 \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(1)}\zeta(2) - \left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^0 \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right)^3 N_1^{(1)}\zeta(3),$$

$$N_{-3}^{(1)} = \frac{1}{4!}\left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^3 N_0^{(1)} - \frac{1}{2!}\left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^2 \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(1)}\zeta(2) - \left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^1 \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right)^3 N_1^{(1)}\zeta(3) - \frac{1}{0!}\left(\frac{N_0^{(1)}}{N_1^{(1)}}\right)^0 \frac{1}{4}\left(\frac{2}{\sqrt{3}}\right)^4 N_1^{(1)}\left(\zeta(4) - \frac{1}{2}\zeta(2)^2\right).$$

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Predicting Analytical Stokes Data from Numerics...

- "Walking down" *second* diagonal, numerics yield analytics...

$$N_{-1}^{(2)} = \frac{1}{2!}\left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^1 N_0^{(2)} - \frac{1}{0!}\left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^0 \frac{2}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(2)} \zeta(2),$$

$$N_{-2}^{(2)} = \frac{1}{3!}\left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^2 N_0^{(2)} - \frac{1}{1!}\left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^1 \frac{2}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(2)} \zeta(2) - \left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^0 \frac{2}{3}\left(\frac{2}{\sqrt{3}}\right)^3 N_1^{(2)} \zeta(3),$$

$$N_{-3}^{(2)} = \frac{1}{4!}\left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^3 N_0^{(2)} - \frac{1}{2!}\left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^2 \frac{2}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(2)} \zeta(2) - \left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^1 \frac{2}{3}\left(\frac{2}{\sqrt{3}}\right)^3 N_1^{(2)} \zeta(3) -$$

$$- \frac{1}{0!}\left(\frac{N_0^{(2)}}{N_1^{(2)}}\right)^0 \frac{2}{4}\left(\frac{2}{\sqrt{3}}\right)^4 N_1^{(2)} \left(\zeta(4) - \frac{2}{2}\zeta(2)^2\right).$$

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Predicting Analytical Stokes Data from Numerics!

- "Walking down" *third* diagonal, numerics yield analytics!

$$N_{-1}^{(3)} = \frac{1}{2!}\left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^1 N_0^{(3)} - \frac{1}{0!}\left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^0 \frac{3}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(3)}\zeta(2),$$

$$N_{-2}^{(3)} = \frac{1}{3!}\left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^2 N_0^{(3)} - \frac{1}{1!}\left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^1 \frac{3}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(3)}\zeta(2) - \left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^0 \frac{3}{3}\left(\frac{2}{\sqrt{3}}\right)^3 N_1^{(3)}\zeta(3),$$

$$N_{-3}^{(3)} = \frac{1}{4!}\left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^3 N_0^{(3)} - \frac{1}{2!}\left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^2 \frac{3}{2}\left(\frac{2}{\sqrt{3}}\right)^2 N_1^{(3)}\zeta(2) - \left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^1 \frac{3}{3}\left(\frac{2}{\sqrt{3}}\right)^3 N_1^{(3)}\zeta(3) -$$
$$- \frac{1}{0!}\left(\frac{N_0^{(3)}}{N_1^{(3)}}\right)^0 \frac{3}{4}\left(\frac{2}{\sqrt{3}}\right)^4 N_1^{(3)}\left(\zeta(4) - \frac{3}{2}\zeta(2)^2\right).$$

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Predicting Analytical Stokes Data from Numerics ✓

- Changing diagonals at fixed $x = 1$:

$$N_1^{(s)} = \frac{\mathrm{i}^{s-1}}{s} \left(N_1^{(1)} \right)^{2-s}.$$

- Infinite set of Stokes data $\Rightarrow$ two numbers:
  - $N_1^{(1)} = -\mathrm{i}\frac{3^{1/4}}{2\sqrt{\pi}}$, **[David]**
  - $N_0^{(1)} = -\frac{\mathrm{i}}{3^{1/4}\sqrt{\pi}} \left(\gamma_{\mathsf{E}} + \log\left(2^5 \cdot 3^{\frac{3}{2}} \right) \right)$. **[Baldino-RS-Schwick-Vega]**

- Allows for conjecture on closed-form asymptotics $\Rightarrow$ leading to (analytical) generating functions for *all* Stokes data!

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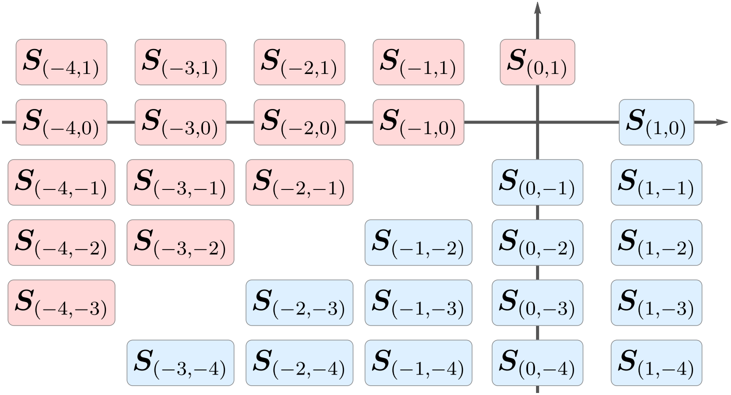
Analytical Stokes Data



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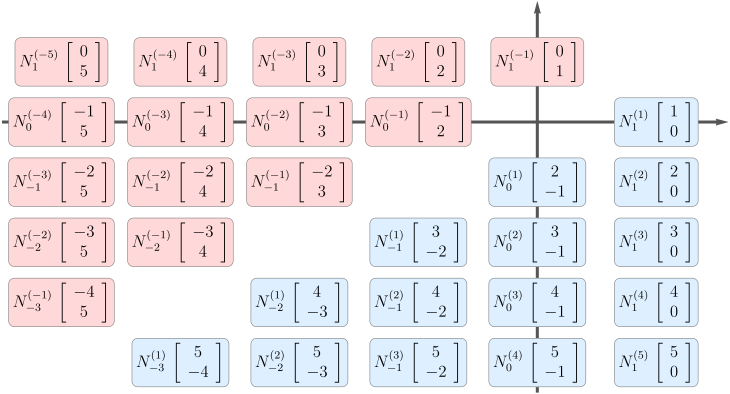
Analytical Stokes Data – Vectorial Structure



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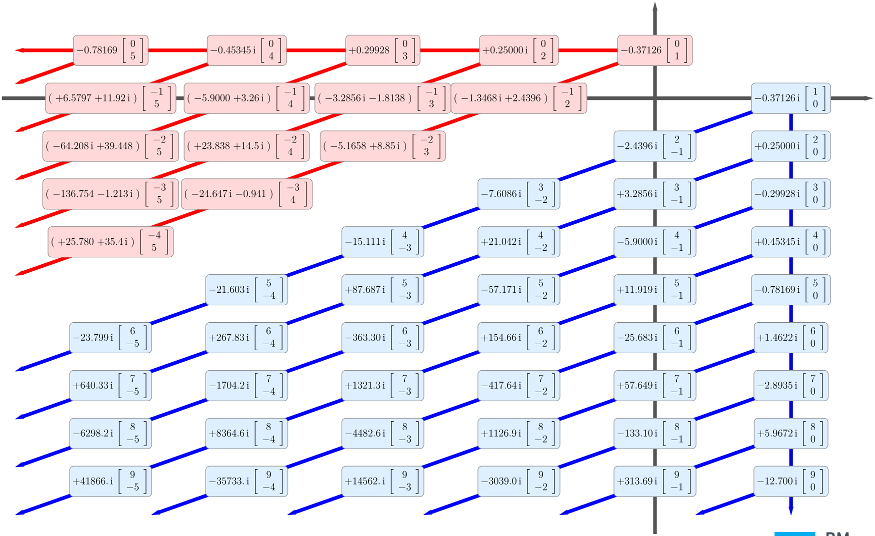
Analytical Stokes Data – Numerology



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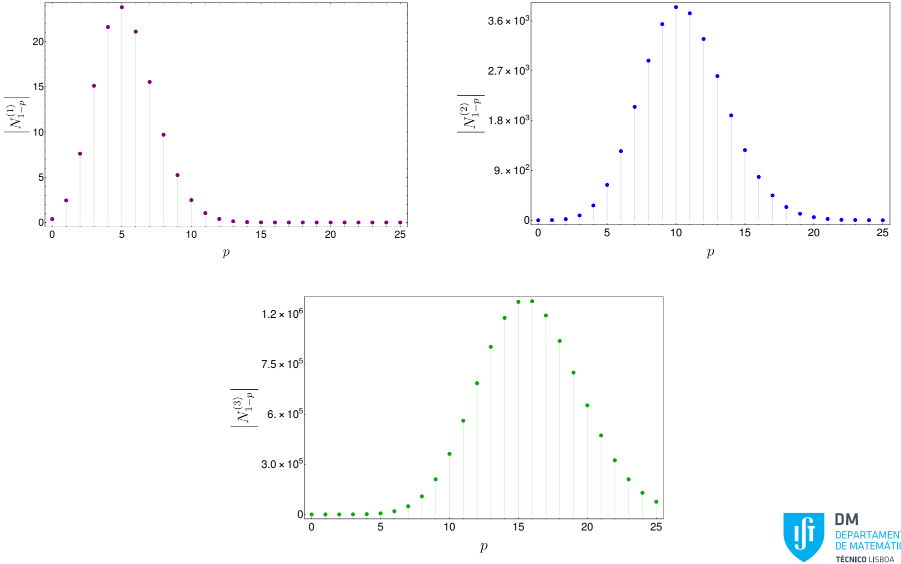
Analytical Stokes Data – Plotting $N^{(s)}_{1-n}$ on Diagonals



ERC Synergy Questions ...??

- *Having full nonlinear Stokes data, can one set up "nonlinear Riemann–Hilbert" problem for Painlevé I solution?...*

- *Would this would-be "nonlinear Riemann–Hilbert" problem have some relation with problem of locating Painlevé I singularities?...*

Help, anyone ...??

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Outline

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Motivation: Nonperturbative 2d Quantum Gravity

- Obtain nonperturbative definition/construction of string theory?
- Obtain semiclassical decoding of this nonperturbative answer? [Mariño]
  - Beyond perturbative $\sim g_s^{\bullet} \Rightarrow$ nonperturbative $\sim \exp\left(-\frac{\bullet}{g_s}\right)$...
- **RE**SURGENT transseries construction ...... requires Stokes data.
  - Nonlinear Stokes data = infinite set of (transcendental) "numbers"...
- Resurgent transseries construction is **RE**SONANT.
- **RE**SOLVENT underlies the resurgent transseries construction.

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Resolvent and Gel'fand–Dikii KdV Potentials

- Hamiltonian for one-dimensional potential $u(x)$ ($\hbar = 1$, $m = \frac{1}{2}$)

$$\mathsf{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x).$$

- Resolvent operator, alongside its integral kernel,

$$\mathsf{R}_\lambda(\mathsf{H}) = \frac{1}{\mathsf{H} + \lambda \mathbf{1}}, \qquad R_\lambda(x,y) = \langle x | \, \mathsf{R}_\lambda(\mathsf{H}) \, | y \rangle.$$

- Diagonal of resolvent yields Gel'fand–Dikii KdV potentials,

$$R_\lambda(x) \simeq \sum_{\ell=0}^{+\infty} \frac{R_\ell\,[u]}{\lambda^{\ell+\frac{1}{2}}} \quad \Rightarrow \quad R'_{\ell+1} = \frac{1}{4} R'''_\ell - u\, R'_\ell - \frac{1}{2} u'\, R_\ell.$$

- $R_\ell\,[u]$ = polynomials in u and its derivatives...

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Gel'fand–Dikii Potentials and String Equations

- Gel'fand–Dikii KdV potentials yield string equations:
  - $k=2$ or $(2,3)$ multicritical theory:

$$R_2 = \frac{1}{16}\left(3u^2 - u''\right) \quad \Rightarrow \quad \frac{16}{3} R_2 = u^2 - \frac{1}{3} u''.$$

    This yields Painlevé I equation $u^2 - \frac{1}{3}u'' = z$ ...
  - $k=3$ or $(2,5)$ multicritical theory:

$$R_3 = -\tfrac{1}{64}\left(10u^3 - 10uu'' - 5\left(u'\right)^2 + u''''\right)$$
$$\Downarrow$$
$$-\tfrac{32}{5} R_3 = u^3 - uu'' - \tfrac{1}{2}\left(u'\right)^2 + \tfrac{1}{10} u''''.$$

    This yields Yang–Lee equation $u^3 - uu'' - \frac{1}{2}\left(u'\right)^2 + \frac{1}{10}u'''' = z$ ...

Gel'fand–Dikii Potentials and Multicritical String Equations

- String equation for $(2, 2k-1)$ theory: [Gross-Migdal]

$$(-1)^k \, \frac{2^{k+1}\, k!}{(2k-1)!!} \, R_k\,[u] = z.$$

- String-theoretic genus expansion ($g_s = z^{-5/4}$):

$$F_{(2,3)} \simeq -\frac{4}{15} z^{\frac{5}{2}} - \frac{1}{24} \log z + \frac{7}{1440} z^{-\frac{5}{2}} + \frac{245}{41472} z^{-5} + \frac{259553}{9953280} z^{-\frac{15}{2}} + \cdots.$$

- String-theoretic genus expansion ($g_s = z^{-7/6}$):

$$F_{(2,5)} \simeq -\frac{9}{28} z^{\frac{7}{3}} - \frac{1}{18} \log z + \frac{1}{120} z^{-\frac{7}{3}} + \frac{247}{27216} z^{-\frac{14}{3}} + \frac{58471}{2099520} z^{-7} + \cdots.$$

- ···

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Nonperturbative Content of Multicritical String Equations?

- Perturbative series are asymptotic $\Rightarrow$ Coefficients grow $F_g \sim (2g)!\ldots$
- String-theoretic nonperturbative effects go as: [Shenker]

$$\sim \exp\left(-\frac{1}{g_{\mathsf{s}}}\right) \equiv \exp\left(-z^{\frac{2k+1}{2k}}\right).$$

- Implies *one*-parameter transseries for order-k multicritical theory:

$$u\left(z,\sigma\right) = z^{\frac{1}{k}} \sum_{n=0}^{+\infty} \sigma^n \exp\left(-nA\, z^{\frac{2k+1}{2k}}\right) z^{-\frac{2k+1}{2k}n\beta} \sum_{g=0}^{+\infty} \frac{u_g^{(n)}}{z^{\frac{2k+1}{2k}g}}.$$

How many parameters?... Resonant?...

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Nonperturbative Content of Painlevé I and Yang–Lee...

- Painlevé I equation / $(2,3)$ multicritical model:
  - Instanton action and characteristic exponent

$$A_{(2,3)} = \pm \frac{4}{5}\sqrt{6}, \qquad \beta_{(2,3)} = \frac{1}{2}.$$

  - One-instanton sector ("plus") expansion

$$u^{(1)}_{(2,3)} \simeq z^{-\frac{1}{8}} \left(1 - \frac{5}{32\sqrt{6}} z^{-\frac{5}{4}} + \frac{75}{4096} z^{-\frac{5}{2}} - \frac{341329}{5898240\sqrt{6}} z^{-\frac{15}{4}} + \cdots \right).$$

- Yang–Lee equation / $(2,5)$ multicritical model:
  - Instanton action and characteristic exponent

$$A_{(2,5)} = \pm \frac{6}{7}\sqrt{5 \pm \mathrm{i}\sqrt{5}}, \qquad \beta_{(2,5)} = \frac{1}{2}.$$

  - One-instanton sector ("plus") expansion

$$u^{(1)}_{(2,5)} \simeq z^{-\frac{1}{4}} \left(1 + \frac{-15 + 8\mathrm{i}\sqrt{5}}{48\sqrt{5 + \mathrm{i}\sqrt{5}}} z^{-\frac{7}{6}} - \frac{197 + 31\mathrm{i}\sqrt{5}}{1024} z^{-\frac{7}{3}} + \cdots \right).$$

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Nonperturbative Content of Multicritical String Equations!

- Instanton action is

$$A = \frac{2k}{2k+1}\,\rho,$$

where ρ is a root of [Ginsparg-ZinnJustin, Gregori-RS]

$$\mathcal{P}_k(\rho) \equiv \sum_{i=1}^{k} \frac{(-1)^i\,\Gamma\left(2k\right)\Gamma\left(k-i+1\right)}{2^{2k}\,\Gamma\left(k\right)\Gamma\left(2k-2i+2\right)\Gamma\left(i\right)}\,\rho^{2k-2i} = 0.$$

- Characteristic exponent is $\beta = \frac{1}{2}$.
  - Degree-$(2k-2)$ polynomial in $\rho \Rightarrow (2k-2)$ instanton actions...
  - Degree-$(k-1)$ polynomial in $\rho^2 \Rightarrow (k-1)$ instanton actions alongside their symmetric pairs...

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Multicritical Resonant Resurgent Transseries

- Order-k multicritical theory: $(2k-2)$-parameter resonant transseries!
- General multi-parameter, resonant transseries solution [Gregori-RS]

$$u\left(z,\boldsymbol{\sigma}\right)=\sum_{\boldsymbol{n}\in\mathbb{N}_0^{2k-2}}\boldsymbol{\sigma}^{\boldsymbol{n}}\exp\left(-\boldsymbol{n}\cdot\boldsymbol{A}\,z^{\frac{2k+1}{2k}}\right)\Phi_{\boldsymbol{n}}(z),$$

with transseries parameters $\boldsymbol{\sigma}^{\boldsymbol{n}}=\prod_{i=1}^{2k-2}\sigma_i^{n_i}$ and instanton "vector" $\boldsymbol{A}=\frac{2k}{2k+1}\left(\rho_1,\cdots,\rho_{k-1},-\rho_1,\cdots,-\rho_{k-1}\right)$.

- Asymptotic sectors have different starting orders $\beta_{\boldsymbol{n}}$

$$\Phi_{\boldsymbol{n}}\simeq\sum_{g=0}^{+\infty}\frac{u_g^{(\boldsymbol{n})}}{z^{\frac{2k+1}{2k}g+\beta_{\boldsymbol{n}}}}.$$

- Resonance: $\mathfrak{P}:\boldsymbol{\ell}\mapsto\boldsymbol{A}\cdot\boldsymbol{\ell}$ has $\ker\mathfrak{P}\neq\boldsymbol{0}$.

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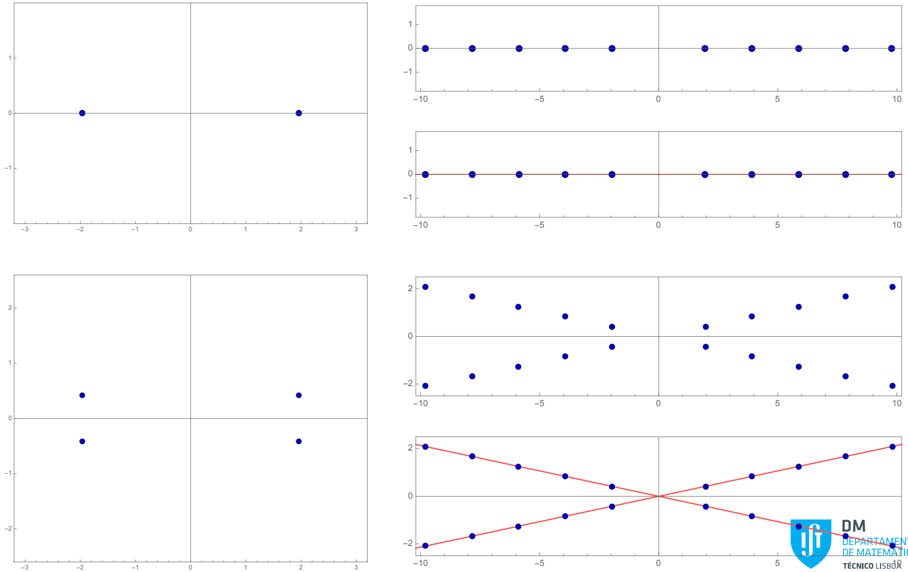
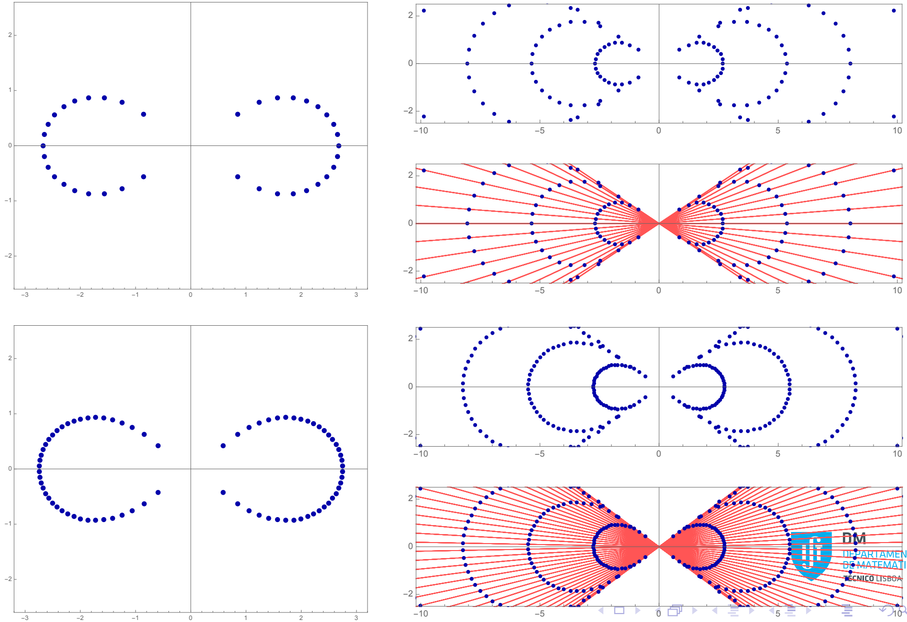
Borel Plane: Painlevé I and Yang–Lee...



Borel Plane: Multicritical Theories $k = 20$ and $k = 41$...



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Outline

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Matrix Models and 't Hooft Large N Limit

- Hermitian one-matrix model with polynomial potential $V(z)$,

$$Z_N = \frac{1}{\mathrm{vol}\,(\mathsf{U}(N))} \int \mathrm{d}M\, \mathrm{e}^{-\frac{1}{g_\mathsf{s}} \mathbb{T}\mathrm{r} V(M)}.$$

- Consider limit $N \to +\infty$ while $t = g_\mathsf{s} N$ fixed ['t Hooft]. Free energy $F = \log Z$ has *asymptotic* genus expansion (with CY dual [Dijkgraaf-Vafa]),

$$F \simeq \sum_{g=0}^{+\infty} F_g(t)\, g_\mathsf{s}^{2g-2}.$$

- One-cut matrix-model spectral curve $y(x)$ is

$$y(x) = M(x) \sqrt{(x-a)\,(x-b)},$$

with moment function

$$M(x) = \oint_{(0)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{V'(1/z)}{1-zx} \frac{1}{\sqrt{(1-az)\,(1-bz)}}.$$

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Nonperturbative Information from Spectral Curve

- Holomorphic effective potential $V'_{\mathsf{h;eff}}(x) = y(x)$, leading at large N,

$$Z_N = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{2\pi} \exp\left(-\frac{1}{g_{\mathsf{s}}} \sum_{i=1}^{N} V_{\mathsf{h;eff}}(\lambda_i) + \cdots \right).$$

- Instanton action

$$A = V_{\mathsf{h;eff}}(x_0) - V_{\mathsf{h;eff}}(b) = \int_b^{x_0} \mathrm{d}x\, y(x),$$

with non-trivial saddle $V'_{\mathsf{h;eff}}(x_0) = 0 \Rightarrow y(x_0) = 0 \Rightarrow M(x_0) = 0$.

- One-loop around one-instanton $F_1^{(1)}$ times Stokes coefficient S_1

[Mariño-RS-Weiss]

$$S_1 \cdot F_1^{(1)} = -\mathrm{i}\, \frac{b-a}{4} \sqrt{\frac{1}{2\pi M'(x_0) \left[(x_0 - a)(x_0 - b) \right]^{\frac{5}{2}}}}.$$

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Multicritical *versus* Minimal Spectral Geometries

- Closed-string backgrounds via KdV times:
  - Multicritical background: lowest dimension operator...
    [DiFrancesco-Ginsparg-ZinnJustin]
  - Conformal background: bulk cosmological constant...
    [Moore-Seiberg-Staudacher, Seiberg-Shih]
- Multicritical-background spectral curve

$$y = -2\sqrt{2}\,\frac{\Gamma\left(k+1\right)}{(2k-1)!!}\sum_{i=0}^{[(k-1)/2]}\binom{(2k-1)/2}{i}T_{2k-1-4i}\big(\sqrt{x/2}\big).$$

- Minimal-string conformal-background spectral curve

$$y^2 = \frac{1}{2}\left(1 + T_{2k-1}(x)\right).$$

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Stokes Data for Multicritical Backgrounds

- Example of $(2,5)$ multicritical model:

$$y_{(2,5)}(x) = \frac{1}{5}\left(8x^2 - 20x + 15\right)\sqrt{x}.$$

- Non-trivial saddles are complex:

$$x_+ = \frac{1}{4}\left(5 + \mathrm{i}\sqrt{5}\right), \qquad x_- = \frac{1}{4}\left(5 - \mathrm{i}\sqrt{5}\right).$$

- Instanton actions from spectral geometry:

$$A_{(2,5)} = -\frac{6}{7}\sqrt{5 \pm \mathrm{i}\sqrt{5}}.$$

- Stokes data from spectral geometry:

$$S_1 \cdot F_1^{(1)} = -\frac{\left(25 + 5\mathrm{i}\sqrt{5}\right)^{\frac{1}{4}}}{2\sqrt{2\pi}\left(-5\mathrm{i} + \sqrt{5}\right)^{\frac{3}{2}}}.$$

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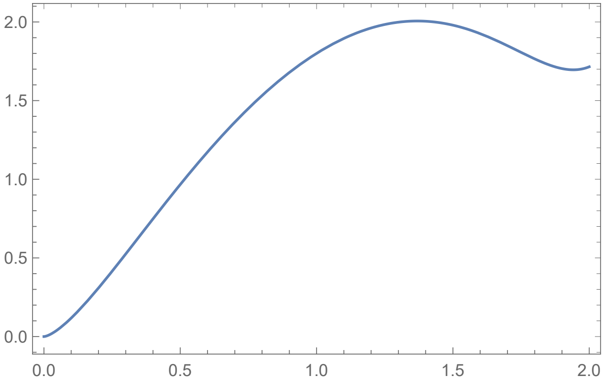
Multicritical Holomorphic Effective Potential



Stokes Data for Conformal Backgrounds

- Example of $(2,5)$ minimal-string model:

$$y=\frac{1}{\sqrt{2}}\left(4x^2-2x-1\right)\sqrt{x+1}.$$

- Non-trivial saddles are real:

$$x_1=-\cos\frac{2\pi}{5},\qquad x_2=-\cos\frac{4\pi}{5}.$$

- Instanton actions from spectral geometry:

$$A_1=-\frac{5}{21}\sqrt{2\left(5+\sqrt{5}\right)},\qquad A_2=\frac{5}{21}\sqrt{2\left(5-\sqrt{5}\right)}.$$

- Stokes data from spectral geometry:

$$S_1\cdot F_1^{(1)}=\frac{\mathrm{i}}{2\left(10-2\sqrt{5}\right)^{\frac{1}{4}}\sqrt{5\left(-5+3\sqrt{5}\right)\pi}}.$$

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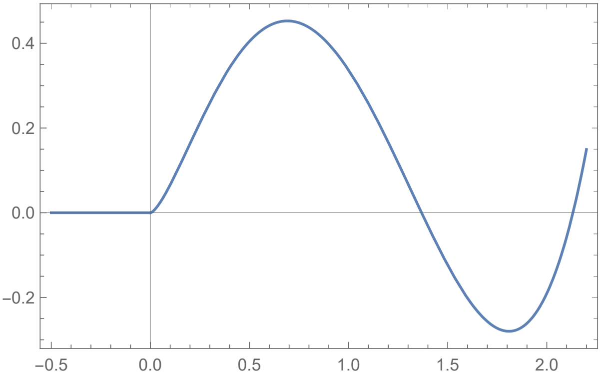
Conformal Holomorphic Effective Potential



ERC Synergy Questions ...??

- *Can compute full nonlinear Stokes data along KdV hierarchy?...*

Probably yes...

- *With full Stokes data, can set up "nonlinear Riemann–Hilbert" problem for generic multicritical/minimal string solutions?...*
- *Would this "nonlinear Riemann–Hilbert" problem relate to problem of locating generic multicritical/minimal string singularities?...*

Help, anyone ...??

- *How generic is resonance?...*

Maybe sometime soon... [Mariño-RS-Schwick]

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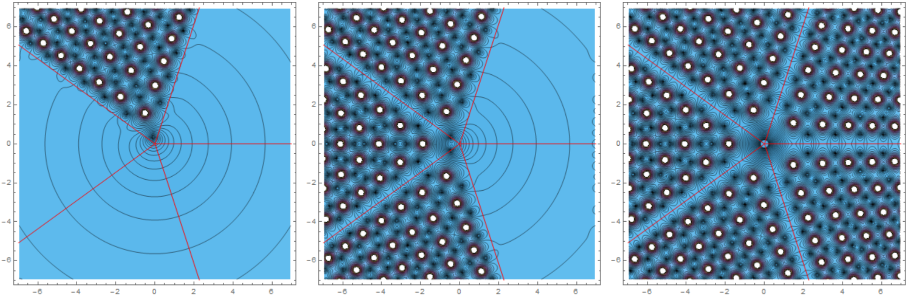
Outline

Comment on Exact Location of Painlevé Poles



- Movable singularities = double-poles. Classification: [Painlevé, Boutroux]
  - Tritronquée solution... Tronquée solution... General solution...
- What is *exact* location of double-poles, as function of initial data?

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Transasymptotic Resummations of Resurgent Transseries

- Resummation methods help describing different phases of our systems.
- Painlevé (multicritical) solutions generically have double poles,

$$u(z)\Big|_{z=z_0} \approx \frac{1}{(z-z_0)^2} + \cdots,$$

which are simple zeroes of partition function $Z(z) \approx (z-z_0) + \cdots$.

- Can reorganize (one-parameter) transseries double-sum:

$$u(z,\sigma) = \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-nAz^{\frac{5}{4}}} \sum_{g=0}^{+\infty} \frac{u_g^{(n)}}{z^{\frac{5}{4}(g+\beta_n)}},$$

summing first over all instanton numbers?

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Linear Transasymptotic Summation

- Starting order for each sector $\beta_n \propto n \Rightarrow$ *Linear*.
- Linear transasymptotic summation (with $\tau = \frac{\sigma}{12\sqrt{z}}\, \mathrm{e}^{-Az^{\frac{5}{4}}}$): [Costin]

$$u(z,\tau) \simeq \frac{1+10\tau+\tau^2}{(1-\tau)^2} + \cdots.$$

- *First* array of (tronquée) poles located at:

$$\tau = 1 \qquad \Leftrightarrow \qquad \frac{\sigma}{12\sqrt{z}}\, \mathrm{e}^{-Az^{\frac{5}{4}}} = 1.$$

- Sub-leading results iteratively yield following arrays of poles...

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Quadratic Transasymptotic Summation

- Linear transasymptotic summation: $\mathbb{Z}_0(\tau) = 1 - \tau$.
- For the partition function, starting order $\beta_n \propto n^2 \Rightarrow$ *Quadratic*.
- Quadratic transasymptotics: first sum leading terms for all sectors in the Z-transseries (much more efficient).
- Quadratic transasymptotic summation: **[Aniceto-RS-Vonk]**

$$Z(\tau, q) \simeq \sum_{n=0}^{+\infty} G_2(n+1)\, \tau^n\, q^{n^2} + \cdots.$$

- At leading order already get **all arrays** of poles (zeroes of Z)!

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Outline

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Jackiw–Teitelboim 2D Quantum Gravity Reloaded

- Reconsider JT dilaton-gravity in AdS_2:

$$S_{\mathsf{JT}} = -\frac{S_0}{4\pi}\int \mathrm{d}^2x\,\sqrt{g}\,R - \frac{1}{2}\int \mathrm{d}^2x\,\sqrt{g}\,\phi\,(R+2)+\cdots.$$

- Natural observables = macroscopic loop operators create (asymptotic) boundaries of (regularized) length $\beta \Rightarrow Z = \mathrm{tr}\,\mathrm{e}^{-\beta\mathsf{H}}$.
- Disk partition function one-loop exact: [Stanford-Witten]

$$Z_{\mathsf{disk}}(\beta) = \mathrm{e}^{S_0}\,\frac{\mathrm{e}^{\pi^2/\beta}}{\sqrt{16\pi\beta^3}}.$$

- Cannot be full story ⇒ Continuous spectrum (from density of states) ⇒ Not holographic... Holography: requires sum over all higher topologies (genus <u>*and*</u> boundaries) ✓ [Saad-Shenker-Stanford]

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JT-Gravity Holographic Perturbative Expansions

- Spacetime $g_{\mathsf{s}} \sim \mathrm{e}^{-S_0}$, holographic matrix model $N \sim \mathrm{e}^{S_0}$.
  [Saad-Shenker-Stanford]
- Sum over all topologies = sum over all genus-g, n-boundary, $Z_{g,n}$-spacetimes:

$$\underbrace{\langle Z(\beta_1) \cdots Z(\beta_n) \rangle}_{n \text{ boundaries}} = \sum_{g=0}^{+\infty} \mathrm{e}^{-(2g-2+n)S_0}\, Z_{g,n}(\beta_1, \ldots, \beta_n).$$

  Generated recursively via Weil–Petersson volumes ✓ [Mirzakhani]
- Holographic matrix model calculation

$$\langle Z(\beta_1) \cdots Z(\beta_n) \rangle_{\mathsf{MM}} = \int \mathrm{d}\mathsf{H}\, \mathrm{e}^{-N \operatorname{tr} V(\mathsf{H})}\ \operatorname{tr} \mathrm{e}^{-\beta_1 \mathsf{H}} \cdots \operatorname{tr} \mathrm{e}^{-\beta_n \mathsf{H}}.$$

  Generated recursively via topological recursion ✓ [Eynard-Orantin]

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From Minimal Towards JT Spectral Geometry

- JT matrix model spectral curve

$$y = \sin\left(2\pi\sqrt{x}\right).$$

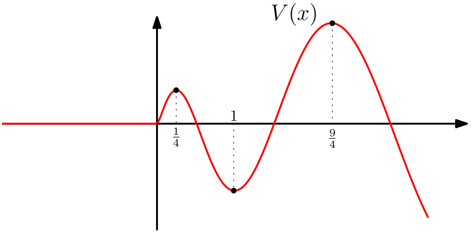
- Minimal-string conformal-background spectral curve at large k

$$\begin{cases} x = T_2\left(\zeta\right) \\ y = T_{2k-1}\left(\frac{2\pi\zeta}{2k-1}\right) \end{cases} \quad \xrightarrow[k\to+\infty]{} \quad y = \sin\left(2\pi\sqrt{x}\right).$$

- Can think of the minimal string as a deformation of JT gravity...?

[Saad-Shenker-Stanford, Seiberg-Stanford, Mertens-Turiaci, Okuyama-Sakai, Johnson, Gregori-RS]

A Plethora of Nonperturbative One-Instanton Sectors



- Infinite instanton sectors, with instanton actions:

$$A_\ell = (-1)^{\ell+1} \frac{\ell}{4\pi^2}.$$

- One-loop around the ℓth one-instanton:

$$\mathcal{F}^{(1)}_{\ell,0} = -\frac{\mathrm{i}^{\ell+1}}{\ell^{3/2}\sqrt{2\pi}}.$$

[Eynard-Failde-Gregori-Lewański-Ooms-RS]

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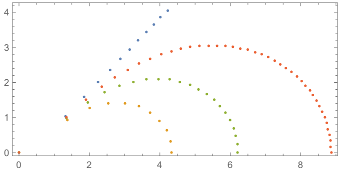
Multicritical Background Large-k Limit as ... Gravity??

- Conformal background is not only background one may consider!
- Multicritical-background spectral curve at large k

$$\begin{cases} x = T_2(\zeta) \\ y = -2\sqrt{2} \sum_{i=1}^{2k-1} \frac{4^{i-1}(-1)^{i+k} k!}{(2i-1)!!(k-i)!} \zeta^{2i-1} \end{cases} \quad \xrightarrow[k\to+\infty]{} \quad y = -2\,\mathrm{D}_+\left(\sqrt{2x}\right),$$

with $\mathrm{D}_+(x) = \mathrm{e}^{-x^2} \int_0^x \mathrm{d}t\, \mathrm{e}^{t^2}$ Dawson function.

- *"Dawson" dilaton gravity...? Enumerative problem...??* **[Gregori-RS]**



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Outline

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Summary and (More) Open Questions

- Wrap-up:
  - Resurgent transseries solution for Painlevé equation...
  - Complete description of Painlevé nonlinear Stokes data...
  - Resurgent transseries solutions for KdV hierarchy...
  - Partial description of KdV nonlinear Stokes data...
  - Transseries summation yields movable singularities (zeroes)...
  - Technology extendable to JT gravity *et alia*...

- Questions:
  - Closed-form expression for full nonperturbative partition functions?
    - Double-scaling limit...?
    - Matrix model...?
    - Topological string theory...?
  - Complete description of resurgence of *all* 2d dilaton gravities?

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