$\mathcal{N} = 1$ Super Topological Recursion

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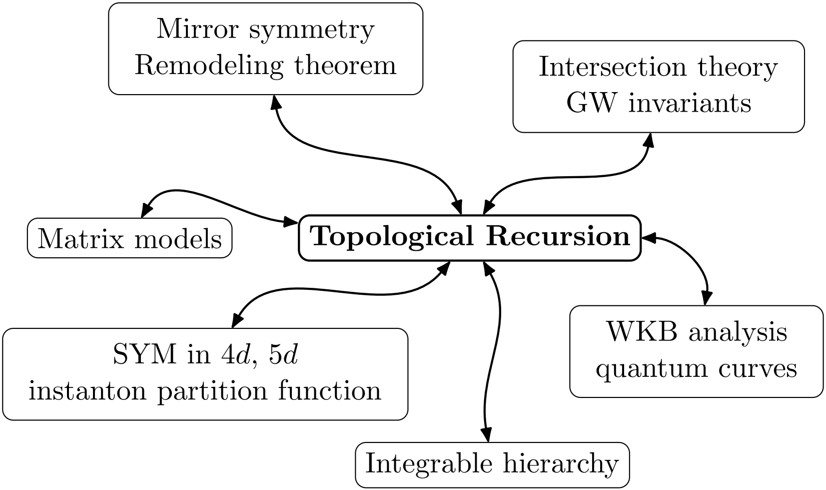
Main References:
1907:08913 with Bouchard, Ciosmak, Hadasz, Ruba, Sułkowski,
2007:13186 with Bouchard,
2107:04588
2207: ? ? ? ? with ? ? ? ?

Intro

Motivation



Q: Are there any fundamental structures underlying?

Story of Topological Recursion

Abstract loop equations

$\swarrow$ $\searrow$

solve geometrically:

- pants decomposition
- residue analysis

$\downarrow$

Topological recursion

solve algebraically:

- vertex operator algebras
- Virasoro constraints

$\downarrow$

Airy structures

Q: Can we extend this story with supersymmetry??

References

Topological recursion and abstract loop equations:

- Chekhov-Eynard (2005, 2006)
- Eynard-Orantin (2007, 2008)
- Borot-Eynard-Orantin (2013)
- Borot-Shadrin (2016)

Airy structures:

- Kontsevich-Soibelman (2017)
- Andersen-Borot-Chekhov-Orantin (2017)
- Bouchard-Borot-Chidambaram-Creutzig-Noshchenko (2018)
- Bouchard-Borot-Chidambaram-Creutzig (2021)

Supersymmetric matrix models:

- Ciosmak-Hadasz-Manabe-Sulkowski (2016, 2017)
- Bouchard-Osuga (2018)
- Osuga (2019)

SAS

Definition

$V = V_0 \oplus V_1 \oplus \mathbb{C}^{0|1}, \quad I_p = \{1, ..., \dim V_p\}, \quad p \in \{0, 1\}$

Coordinates: $((x^i)_{i \in I_0}, (\theta^i)_{i \in I_1}, \theta^0)$

Grading: $\deg x^i = 1 = \deg \theta^i = \deg \theta^0$, $\deg \hbar = 2$.

$\mathcal{D}_\hbar(V) := \mathbb{C}[(x^i)_{i \in I_0}, (\hbar \partial_{x^i})_{i \in I_0}, \hbar, (\theta^i)_{i \in I_1}, \theta^0, (\hbar \partial_{\theta^i})_{i \in I_1}, \hbar \partial_{\theta^0}]$.

Definition: A "super Airy structure" is a set $\mathcal{S}_A = \{L_i^{(p)}\}_{i \in I_p}$ of parity-preserving operators $L_i^{(p)} \in \mathcal{D}_\hbar(V)$ satisfying:

1. $\forall i \in I_0, \quad L_i^{(0)} = \hbar \partial_{x^i} + \deg \geq 2,$
   $\forall i \in I_1, \quad L_i^{(1)} = \hbar \partial_{\theta^i} + \deg \geq 2,$
2. $\forall i, j, k \in I_p$, there exists $f_{ij}^k \in \mathcal{D}_\hbar(V)$ such that
$$[L_i^{(p_i)}, L_j^{(p_j)}]_s = \hbar f_{ij}^k L_k^{(p_k)}.$$

Remarks:

- There is no $L_0^{(1)} = \hbar \partial_{\theta^0} + \cdots$! ($\theta^0$ appears in $\deg \geq 2$ terms)
- \# variables = \# differential operators +1
- we call θ^0 the *extra variable*.

Q: What is it good for?

Partition Function and Free Energy

Theorem:(Bouchard-Ciosmak-Hadasz-O-Ruba-Sułkowski)
Given a super Airy structure $\mathcal{S}_A$, there exists a unique power series $F \in \mathbb{C}[\![(x^i)_{i\in I_0}, (\theta^i)_{i\in I_1}, \theta^0, \hbar]\!]$ such that:

- S1 $F(x,\theta)$ is even (bosonic)
- S2 $F(x,\theta)$ does not have terms of degree ≤ 2,
- S3 $F(x,\theta)$ satisfies the following set of differential equations

$$\forall i \in I_p, \quad L_i^{(p)} e^{\frac{F(x,\theta)}{\hbar}} = 0$$

More explicitly,

$$F(x,\theta) = \sum_{g\geq 0, n\geq 0, m\geq 0} \sum_{i_1,...,i_n \in \{I_0\}} \sum_{j_1,...,j_m \in \{0, I_1\}} \frac{\hbar^g}{n!m!} F_{g,n|m}(i_1,...,i_n|j_1,...,j_m) x^{i_1}\cdots x^{i_n}\theta^{j_1}\cdots\theta^{j_m}$$

By assumption, $F_{g,n|m}(i_1,...,i_n|j_1,...,j_{2m}) \in \mathbb{C}$
S1 & S2 implies $F_{g,n|2m+1} = F_{0,1|0} = F_{0,2|0} = F_{0,0|2} = 0$
S3 gives a set of recursive equations for $F_{g,n|2m}$

Q: How does the recursion look like?

Recursion

Let $\dim V_0 > 0$ and $\dim V_1 = 0$, then consider a *quadratic* Airy structure $\mathcal{S}_A = \{L_i^{(0)}\}_{i \in I_0}$ of the form:

$$L_i^{(0)} = \hbar \partial_{x^i} - \frac{1}{2} A_{ijk} x^j x^k - \hbar B_{ij}^k x^j \partial_{x^k} - \frac{\hbar^2}{2} C_i^{jk} \partial_{x^j} \partial_{x^k} - \hbar D_i,$$

where $A_{ijk}, B_{ij}^k, C_i^{jk} D_i \in \mathbb{C}$.

For $2g + n \geq 2$ and $J = \{i_1, ..., i_n\}$, $L_i^{(0)} e^{\frac{F(x)}{\hbar}} = 0$ gives

$$F_{0,3}(i_0, i_1, i_2) = A_{i_0 i_1 i_2}, \quad F_{1,1}(i_0) = D_{i_0},$$

$$F_{g,n+1}(i_0, J) = \sum_{r \in I_0} B_{i_0 i_r}^k F_{g,n}(k, J \backslash i_r) + C_{i_0}^{jk} F_{g-1,n+2}(j, k, J)$$

$$+ \sum_{J_1 \cup J_2 = J} \sum_{g_1 + g_2 = g} C_{i_0}^{jk} F_{g_1, n_1+1}(j, J_1) F_{g_2, n_2+1}(k, J_2)$$

- RHS depends only on $F_{h,s}$ of $2h + s < 2g + n + 1$,
- pictorially, RHS is a trivalent-graph decomposition
- higher degrees $\rightarrow$ mutli-valent graph decomposition

Q: Now what happens with fermions?

Recursion with Fermions

For $\dim V_p > 0$, a quadratic super Airy structure $\mathcal{S}_A = \{L_i^{(p)}\}_{i \in I_p}$ is written in the form:

$$L_i^{(0)} = \hbar\partial_{x^i} - \frac{1}{2}A_{ijk}x^j x^k - \hbar B_{ij}^k x^j \partial_{x^k} - \frac{\hbar^2}{2}C_i^{jk}\partial_{x^j}\partial_{x^k} - \hbar D_i$$
$$-\frac{1}{2}E_{i|\alpha\beta}\theta^\alpha\theta^\beta - \hbar F_{i|\alpha}^{\beta}\theta^\alpha\partial_{\theta^\beta} - \frac{\hbar^2}{2}G_{i|}^{\alpha\beta}\partial_{\theta^\alpha}\partial_{\theta^\beta}$$

where $A_{ijk}, B_{ij}^k, C_i^{jk}, D_i, E_{i|\alpha\beta}, F_{i|\alpha}^{\beta}, G_{i|}^{\alpha\beta} \in \mathbb{C}$, and $\alpha, \beta \in \{0, I_1\}$.

- Similarly, $\{L_i^{(1)}\}_{i \in I_1}$ are written with four more parameters $H_{j|\alpha\beta}, I_{j|\alpha}^{\beta}, J_{|\alpha\beta}^{j}, K_{|\alpha}^{j\beta} \in \mathbb{C}$
- $L_i^{(p)} e^{\frac{F(x,\theta)}{\hbar}} = 0$ gives a recursion for $F_{g,n|2m}$ in terms of $ABCDEFGHIJK$.
- trivalent graph decomposition with fermionic edges (and special care for signs and the extra variable)

Q: How can we find *interesting* super Airy structures?

VOA

Vertex Operator Algebras

Airy structures for TR of ramification order r	$\leftrightarrow$	twisted modules of $\mathcal{W}(\mathfrak{gl}_r)$-algebra
Airy structures for TR *without branch covers*	$\leftrightarrow$	untwisted modules of $\mathcal{W}(\mathfrak{gl}_r)$-algebra

4 classes of super Airy structures can be constructed in terms of modules of the $\mathcal{N}=1$ super Virasoro algebra:

classes	boson	fermion	sector	application
untwisted	$\times$	$\times$	NS	$\bigcirc$
μ-twisted	$\times$	$\bigcirc$	R	$\bigcirc$
σ-twisted	$\bigcirc$	$\times$	R	?
ρ-twisted	$\bigcirc$	$\bigcirc$	NS	$\bigcirc$

ρ-twisted module turns out to be a natural supersymmetric generalisation of topological recursion of simple ramification

Q: What is the ρ-twisted module?

ρ-twisted module

For $\{x^1, x^2, ...\}$ and $\{\theta^0, \theta^1, \theta^2, ...\}$,

$$\forall a \in \mathbb{Z}_{>0}, \quad J_a = \hbar \partial_{x^a}, \quad J_0 = 0, \quad J_{-a} = a x^a,$$

$$\forall a \in \mathbb{Z}_{>0}, \quad \Gamma_a = \hbar \partial_{\theta^a}, \quad \Gamma_0 = \frac{\theta^0}{2} + \hbar \partial_{\theta^0}, \quad \Gamma_{-a} = \theta^a,$$

$$n \in \mathbb{Z}, \quad L_{2n} = \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^{j-1} : J_{-j} J_{2n+j} : + \frac{\hbar}{4} \delta_{n,0} + \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^j (n+j) : \Gamma_{-j} \Gamma_{j+2n} :$$

$$m \in \mathbb{Z}, \quad G_{2m+1} = \sum_{j \in \mathbb{Z}} (-1)^{j-1} : J_{-j} \Gamma_{j+2m+1} :$$

$\{J_{2i}, \Gamma_{2i+1}, L_{2i}, G_{2i+1}\}_{i \in \mathbb{Z}}$ generates (a slight extension of) the $\mathcal{N} = 1$ super Viraroso algebra in the Neveu-Schwarz sector.
L_{2i}, G_{2i+1} do not have deg 1 terms $(= \hbar \partial_{x^i}, \hbar \partial_{\theta^i})$!?

Q: How to turn it into a super Airy structure?

Conjugation – adding degrees of freedom

For $\tau_l, \phi_{kl}, \psi_{\alpha\beta} \in \mathbb{C}$ with $k, l \geq 1$, $\alpha, \beta \geq 0$,
$(|\tau_1| = 0,\ |\tau_3| \neq 0\ \phi_{kl} = \phi_{lk},\ \psi_{\alpha\beta} + \psi_{\beta\alpha} + \psi_{\alpha 0}\psi_{\beta 0} = 0,\ \psi_{00} = 0)$,

$$\Phi := \exp\left(\frac{1}{\hbar}\left(\sum_{l>0}\frac{\tau_l}{l}J_l + \sum_{l,k>0}\frac{\phi_{kl}}{2kl}J_kJ_l + \sum_{\alpha,\beta\geq 0}\frac{\psi_{\alpha\beta}}{2}\Gamma_\alpha\Gamma_\beta\right)\right)$$

$$\forall i \geq 1, \quad H_i^1 = \Phi J_{2i}\Phi^{-1}, \quad H_i^2 = \Phi L_{2i-4}\Phi^{-1},$$
$$F_i^1 = \Phi\Gamma_{2i-1}\Phi^{-1}, \quad F_i^2 = \Phi G_{2i-3}\Phi^{-1}.$$

Proposition:(Bouchard-O)
$\mathcal{S}_A = \{H_i^1, H_i^2, F_i^1, F_i^2\}_{i\geq 1}$ defines a super Airy structure for every choice of $\tau_l, \phi_{kl}, \psi_{ab}$.
$\Rightarrow$ there exists a unique free energy F.

$$\boxed{\text{SAS} \ :\ \tau_l, \phi_{kl}, \psi_{ab} \ \longrightarrow\ F_{g,n|2m}(i_1, ..., i_n|j_1, ..., j_{2m})}$$

Q: How can we find abstract super loop equations?

STR

Story of Super Topological Recursion?

Abstract *super* loop equations?

↙ ↘

solve geometrically:

- pants decomposition
- residue analysis

↓

Super Topological recursion?

solve algebraically:

- vertex operator algebras
- super Virasoro constraints

↓

Super Airy structures

Virasoro algebras → $\mathcal{N} = 1$ super Virasoro algebras but...

Q: Any constructive approach?

Things to Check

Hermitian matrix models $\to$ topological recursion
supersymmetric analogue of Hermitian matrix models?

- supereigenvalue models in the NS/R sector
  - derive their loop equations for correlation functions
  - obtain a polynomial equation in the leading order
  - investigate pole structures of correlation functions
- find a universal structure in their loop equations, regardless of the sector (highly nontrivial)
- define abstract super loop equations
- solve them algebraically by super Airy structures and geometrically by super topological recursion

Q: What is the initial data of the recursion?

Super Spectral Curve

Definition:

A local super spectral curve $\mathcal{S}_C$ is a quintuple
$\mathcal{S}_C = (\mathbb{C}^{1|1}, \sigma, \omega_{0,1|0}, \omega_{0,2|0}, \omega_{0,0|2})$

- $\sigma : (z, \theta) \mapsto (-z, \theta)$,
- a choice of τ_l

$$\omega_{0,1|0}(z) = \sum_{l>0} \tau_l z^{l-1} dz,$$

- a choince of ϕ_{kl}

$$\omega_{0,2|0}(z_1, z_2|) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \sum_{k,l>0} \phi_{kl}\ z_1^{k-1} z_2^{l-1} dz_1 dz_2,$$

- a choice of ψ_{ab} with $\Theta = \theta + z dz \partial_\theta$

$$\omega_{0,0|2}(|z_1, z_2) = -\frac{1}{2} \frac{z_1 + z_2}{z_1 - z_2} \frac{\Theta_1 \Theta_2}{z_1 z_2} - \frac{1}{2} \sum_{a,b \geq 0} \frac{\psi_{ab} - \psi_{ba}}{1 + \delta_{ab,0}} z_1^{b-1} z_2^{a-1} \Theta_1 \Theta_2$$

$(|\tau_1| = 0,\ |\tau_3| \neq 0\ \phi_{kl} = \phi_{lk},\ \psi_{\alpha\beta} + \psi_{\beta\alpha} + \psi_{\alpha 0}\psi_{\beta 0} = 0,\ \psi_{00} = 0)$

Q: Why is $\omega_{0,0|2}$ in this form?

A Few Remarks

Remarks:

- $(\tau_l, \phi_{kl}, \psi_{ab})$ completely fixes a local super spectral curve
- $\Theta^2 = zdz$ so $\Theta \sim \sqrt{zdz}$,
- in the limit $z_1 \to z_2$,

$$\omega_{0,0|2}(|z_1, z_2) \sim \frac{\sqrt{dz_1\, dz_2}}{z_1 - z_2} + \cdots .$$

Given a local super spectral curve $\mathcal{S}_C$, we can then define *abstract super loop equations* for $\omega_{g,n|2m}$
There is a set of formulae that recursively solves abstract super loop equations, and construct $\omega_{g,n|2m}(z_1, ..., z_n|u_1, ..., u_{2m})$ for $2g + n + 2m \geq 3$ ($u_i = (u_i, \theta_i)$).
We call the set of these recursive formulae $\omega_{g,n|2m}$

"$\mathcal{N} = 1$ super topological recursion"

Q: How different is it from the standard one?

Super Topological Recursion

For $J = (z_1, ..., z_n)$, $K = (u_1, ..., u_{2m})$

$$\omega_{g,n+1|2m}(z_0, J|K) = \frac{1}{2}\operatorname*{Res}_{z\to 0} K^{BB}(z_0, z, \sigma(z))\Bigg($$

$$\omega_{g-1,n+2|2m}(z, \sigma(z), J|K)$$

$$+ \sum_{g_1+g_2=g} \sum_{\substack{J_1\cup J_2=J \\ K_1\cup K_2=K}} (-1)^{\rho}\omega_{g_1,n_1+1|2m_1}(z, J_1|K_1)\,\omega_{g_2,n_2+1|2m_2}(\sigma(z), J_2|K_2)$$

$$- \mathcal{D}_z \cdot \omega_{g-1,n|2m+2}(J|z, u, K)\Big|_{u=\sigma(z)}$$

$$+ \sum_{g_1+g_2=g} \sum_{\substack{J_1\cup J_2=J \\ K_1\cup K_2=K}} (-1)^{\rho}\mathcal{D}_z \cdot \omega_{g_1,n_1|2m_1}(J_1|z, K_1)\,\omega_{g_2,n_2|2m_2}(J_2|\sigma(z), K_2)$$

$$+ \Big(z \leftrightarrow \sigma(z)\Big)\Bigg)$$

gluing two fermionic edges!

There is a formula for $\omega_{g,n|2m}(J|u_1, \tilde{K})$ with $\tilde{K} = (u_2, ..., u_{2m})$ too.

Q: What is a relation to the super Airy structure?

Equivalence

Summary:

$$\begin{aligned} \text{STR} &: \tau_l, \phi_{kl}, \psi_{ab} \longrightarrow \omega_{g,n|2m}(z_1, ..., z_n|u_1, ..., u_{2m}) \\ \text{SAS} &: \tau_l, \phi_{kl}, \psi_{ab} \longrightarrow F_{g,n|2m}(i_1, ..., i_n|j_1, ..., j_{2m}) \end{aligned}$$

Q: Any relation between $F_{g,n|2m}$ and $\omega_{g,n|2m}$?

Theorem: (Bouchard-O)
For each choice of $(\tau_l, \phi_{kl}, \psi_{ab})$, there exists a natural basis of expansion $(d\xi_{-i})_{i>0}, (\eta_{-j})_{j\geq 0}$ such that

$$\omega_{g,n|2m}(z_1, ..., z_n|u_1, ..., u_{2m}) = \sum_{\substack{i_1,...,i_n>1 \\ j_1,...,j_{2m}\geq 0}} F_{g,n|2m}(i_1, ..., i_n|j_1, ..., j_{2m}) \bigotimes_{k=1}^{n} d\xi_{-i_k}(z_k) \otimes \bigotimes_{l=1}^{2m} \eta_{-j_l}(u_l, \theta_l)$$

Key of the proof : abstract super loop equations!

Q: Any interesting applications in physics and mathematics?

Applications

TR v.s. STR

$$\text{TR with } (\tau_l, \phi_{kl}) \longrightarrow \omega^{\text{TR}}_{g,n}$$

$$\text{STR with } (\tau_l, \phi_{kl}, \psi_{ab}) \longrightarrow \omega_{g,n|2m}$$

Any relations between $\omega^{\text{TR}}_{g,n}$ and $\omega_{g,n|0}$??

- For any $(\tau_l, \phi_{kl}, \psi_{ab})$,
  $\omega_{0,n|0} = \omega^{\text{TR}}_{0,n}$, but $\omega_{g,n|0} \neq \omega^{\text{TR}}_{g,n}$ for $g \geq 1$
- For $(\tau_l, \phi_{kl}, \psi_{ab}) = (\tau_l, 0, 0)$,
  for all $g \geq 0$, $\boxed{\omega_{g,n|0} = 2^g\, \omega^{\text{TR}}_{g,n}}$, $\boxed{\omega_{g,n|2} \neq 0}$, $\boxed{\omega_{g,n|4} = 0}$.

Applications:

1. $(\tau_l, \phi_{kl}, \psi_{ab}) = (\delta_{l,3}, 0, 0)$
Physics: $(2, 4k)$-minimal model coupled to Liouville supergravity
Q: any interpretation for $\omega_{g,n|2}$ in enumerative geometry?

2. $(\tau_l, \phi_{kl}, \psi_{ab}) = (\tau_l, 0, 0)$ with $\sqrt{2}\cos(2\pi z) = \sum_{l\geq 1} \tau_l z^{l-1}$
$\omega_{g,n|0} \to$ volumes of moduli spaces of super Riemann surfaces
Q: any interpretation for $\omega_{g,n|2}$??

Supereigenvalue Models

Supereigenvalue models
= supersymmetric analogues of Hermitian matrix models

The Neveu-Schwarz sector ($\phi_{kl} \neq 0$, $\psi_{ab} \neq 0$)

- (global) bosonic spectral curve: $y^2 - x^2 + t_{\mathrm{NS}} = 0$
- $\omega^{\mathrm{NS}}_{g,n|0} = 2^g\,\omega^{\mathrm{TR}}_{g,n} = 2^g\,\omega^{\mathrm{HMM}}_{g,n}$

A surprise appears in the Ramond sector

The Ramond sector ($\phi_{kl} \neq 0$, $\psi_{ab} \neq 0$)

- (global) bosonic spectral curve: $xy^2 - x + t_{\mathrm{R}} = 0$
- $\omega^{\mathrm{R}}_{g,n|0} \neq 2^g\,\omega^{\mathrm{TR}}_{g,n}$ for $g \geq 1$
- contributions from the Airy-like branch point ($x = t_R$) are computed by the super topological recursion
- there is no contribution from the Bessel-like branch point ($x = 0$) due to supersymmetric effects!

Relation to the AGT correspondence

The following three are mutually related:

- Nekrasov partition functions and Gaiotto vectors for (pure) $\mathcal{N} = 2$ supersymmetric gauge theory ($4d$)
- Whittaker vectors in conformal blocks ($2d$)
- topological recursion *without branch covers*

Can we extend this relation to STR? The answer is, YES

- super topological recursion "without branch covers"
  := super topological recursion corresponding to untwisted and μ-twisted modules
- Whittaker vectors in superconformal blocks ($2d$)
- Nekrasov partition functions and the Gaiotto vectors for (pure) $U(2)$ $\mathcal{N} = 2$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_2$ ($4d$)
  – still conjectural

Summary

choice of $(\tau_l, \phi_{kl}, \psi_{ab})$
$\rightarrow$ a local super spectral curve
$\rightarrow$ a super Airy structure

$$\begin{array}{rcl} \text{SAS} : \tau_l, \phi_{kl}, \psi_{ab} & \longrightarrow & F_{g,n|2m}(i_1, ..., i_n | j_1, ..., j_{2m}) \\ \text{STR} : \tau_l, \phi_{kl}, \psi_{ab} & \longrightarrow & \omega_{g,n|2m}(z_1, ..., z_n | u_1, ..., u_{2m}) \end{array}$$

Many more to be investigated...

- enumerative interpretations of $\omega_{g,n|2m\geq 2}$
- extension to $\mathcal{W}(\mathfrak{osp}(1|2n))$-algebras?
- $\mathcal{N} = 2$ super Virasoro algebra?
- global super spectral curves?
- Super quantum curves?
- Super CohFT?

Thank you