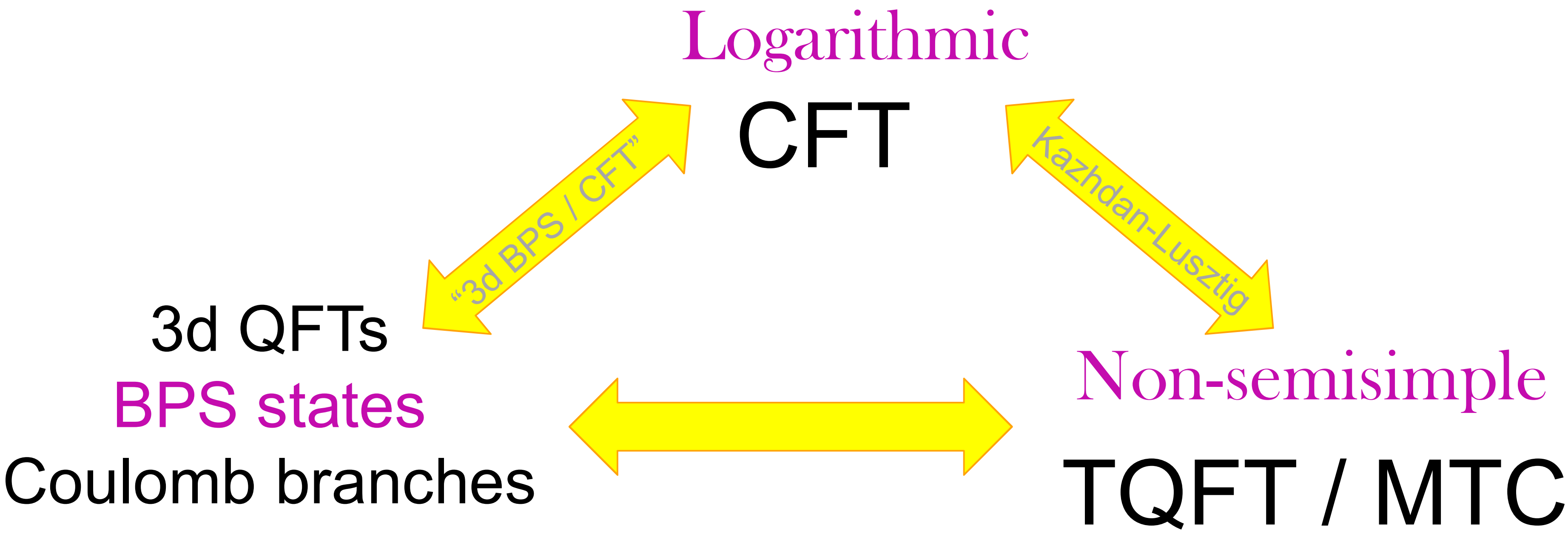
Non-semisimple TQFT's and BPS q-series

FRANCESCO COSTANTINO, SERGEI GUKOV, AND PAVEL PUTROV



J.Murakami
Y.Akutsu, T.Deguchi, T.Ohtsuki
E.Date, M.Jimbo, K.Miki, T.Miwa
T.Deguchi
V.Lyubashenko
D.De Wit, A.Ishii, J.Links
R.Kashaev, N.Reshetikhin
J.Murakami, K.Nagatomo
N.Geer, B.Patureau-Mirand, V.Turaev
F.Costantino, N.Geer, B.Patureau-Mirand
T.Creutzig, A.Milas, M.Rupert
A.Beliakova, K.Hikami
M.De Renzi, A.Gainutdinov, N.Geer, B.Patureau-Mirand, I.Runkel

During last year:

S.Park
S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko
J.Brown, T.Dimofte, S.Garoufalidis, N.Geer
J.Chae
F.Ferrari, P.Putrov
J.Qiu
T.Creutzig, T.Dimofte, N.Garner, N.Geer
B.Feigin, S.G., N.Reshetikhin

Based on the spectacular success of the Khovanov homology, that categorifies the Jones polynomial,

$$J_K(q) = \sum_{i,j} (-1)^i q^j \, \dim Kh_{i,j}(K)$$

it is natural to ask whether Witten-Reshetikhin-Turaev (WRT) invariants of 3-manifolds admit a similar categorification:

$$\mathrm{WRT}(M_3; k) = \sum \, \ldots \, \dim H(M_3)$$

One immediate obstacle is that the WRT invariants, defined at roots of unity, do not come in the form of a polynomial / power series in $q = \exp(2\pi i/k)$ with integer coefficients, e.g.

$$\left(\frac{k}{2}\right)^{g-1} \sum_{j=1}^{k-1} \left(\sin \frac{\pi j}{k}\right)^{2-2g}$$

Possible ways around this challenge:

- Hopfological algebra — M.Khovanov, Y.Qi, A.Beliakova, ...
- Higher representation theory — R.Rouquier, A.Manion, ...
- Holomorphic q-series in |q|<1 — this talk

Surprise: multiple q-series

$$\widehat{Z}_b(M_3;q) = \sum_{i,j} (-1)^i q^j \dim H^{i,j}(M_3;b)$$

S.G., P.Putrov, C.Vafa
S.G., M.Marino, P.Putrov

labeled by $b \in H_1(M_3;\mathbb{Z}) \cong \mathrm{Spin}^c(M_3)$

S.G., C.Manolescu
S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

so that $\mathrm{WRT}(M_3,k) = \sum_b c_b^{\mathrm{WRT}} \widehat{Z}_b(q)\Big|_{q\to e^{\frac{2\pi i}{k}}}$

First clues:

$$\mathrm{WRT}(M_3,k) = \sum_b c_b^{\mathrm{WRT}} \widehat{Z}_b(q)\Big|_{q\to e^{\frac{2\pi i}{k}}}$$

$$= \sum_{a,b} e^{2\pi i k \mathrm{CS}(a)} S_{ab} \widehat{Z}_b(q)\Big|_{q\to e^{\frac{2\pi i}{k}}}$$

S-matrix of a non-semisimple (logarithmic) MTC

S.G., P.Putrov, C.Vafa
S.G., D.Pei, P.Putrov, C.Vafa

<u>Conj (*"3d Modularity"*):</u>

for any M_3 $\exists$ log-VOA$[M_3]$ s.t. $\widehat{Z}_b(M_3) = \chi_b$

characters of a log-VOA $\forall\, b$

K.Bringmann, K.Mahlburg, A.Milas
M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison

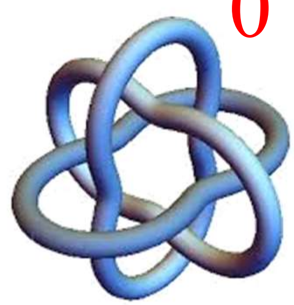
In the approach based on surgeries, one first needs to construct invariants for knot (or link) complements:

$$F_K(x,q) := \sum_{b \in \mathbb{Z}} x^b \, \widehat{Z}_b \left(S^3 \setminus K \right)$$

Theorem [Lickorish, Wallace]:

Every connected oriented closed 3-manifold arises by performing an integral Dehn surgery along a link in S^3.

$$T^3 =$$



In the approach based on surgeries, one first needs to construct invariants for knot (or link) complements:

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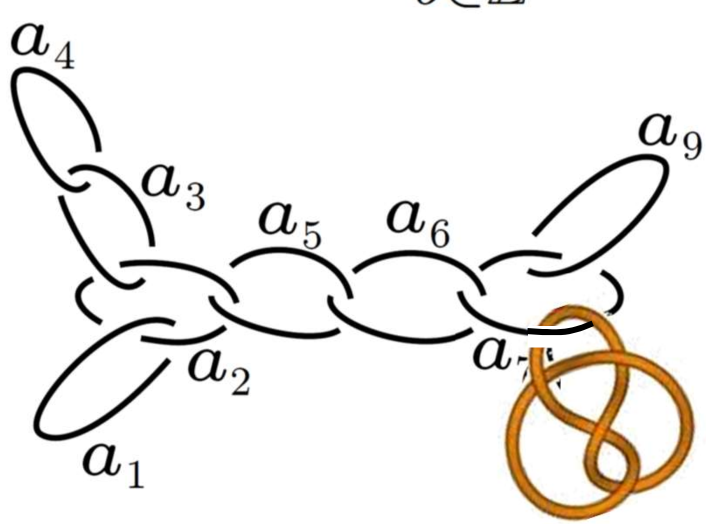


Surgery formula:

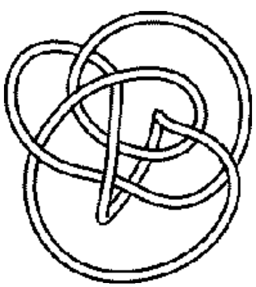
S.G., C.Manolescu

$$\widehat{Z} = \sum_{n_v} \oint_{|z_v|=1} \frac{dz_v}{2\pi i z_v} \prod_{\text{vertices}} (\ldots) \prod_{\text{edges}} (\ldots)$$

For knot and link complements, a very efficient diagrammatic approach based on the R-matrix for Verma modules and quantum groups at generic q was proposed by S. Park (2020, 2021).

For example, using this approach and the GM surgery formula one finds:

$S^3_{+5}(\mathbf{10_{145}})$

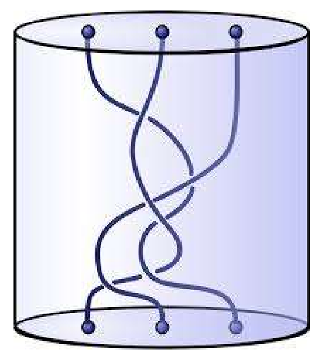


$$
\begin{aligned}
b = 2 :& \quad q^{14/5}\left(-1 + 2q + 2q^2 + q^3 + \ldots\right) \\
b = 1 :& \quad q^{11/5}\left(-1 - 2q^2 - 2q^3 - 4q^4 + \ldots\right) \\
b = 0 :& \quad 2q^4 + 2q^7 + 2q^8 + 2q^9 + 4q^{10} + \ldots \\
b = -1 :& \quad q^{11/5}\left(-1 - 2q^2 - 2q^3 - 4q^4 + \ldots\right) \\
b = -2 :& \quad q^{14/5}\left(-1 + 2q + 2q^2 + q^3 + \ldots\right)
\end{aligned}
$$

arXiv:2004.02087 5 Apr 2020

LARGE COLOR R-MATRIX FOR KNOT COMPLEMENTS AND STRANGE IDENTITIES

SUNGHYUK PARK



$$\check{R}(v^i \otimes w^j) = q^{\frac{nm-1}{4}} x^{-\frac{1}{4}} y^{-\frac{1}{4}}$$
$$\times \sum_{k\geq 0} \begin{bmatrix} i \\ k \end{bmatrix} \prod_{1\leq l\leq k} \left(1 - y^{-1} q^{j+l}\right) x^{-\frac{j}{2}-\frac{k}{4}} y^{-\frac{i-k}{2}+\frac{k}{4}} q^{(i-k)j+\frac{(i-k)k}{2}+\frac{(i-k)+j+1}{2}} w^{j+k} \otimes v^{i-k}$$

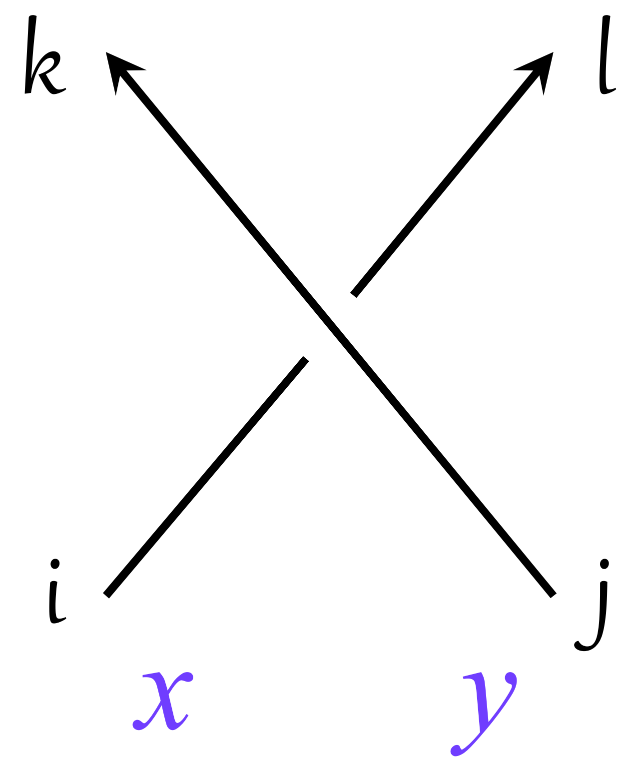
LARGE COLOR R-MATRIX FOR KNOT COMPLEMENTS AND STRANGE IDENTITIES

SUNGHYUK PARK

generic $|q|<1$

Verma modules
(complex weights)

Baxterization?
quasiparticles?



$$\check{R}(v^i \otimes w^j) = q^{\frac{nm-1}{4}} x^{-\frac{1}{4}} y^{-\frac{1}{4}} \times \sum_{k\geq 0} \begin{bmatrix} i \\ k \end{bmatrix} \prod_{1\leq l\leq k} (1-y^{-1}q^{j+l}) x^{-\frac{j}{2}-\frac{k}{4}} y^{-\frac{i-k}{2}+\frac{k}{4}} q^{(i-k)j+\frac{(i-k)k}{2}+\frac{(i-k)+j+1}{2}} w^{j+k} \otimes v^{i-k}$$

Journal of Knot Theory and Its Ramifications, Vol. 1 No. 2 (1992) 161–184

INVARIANTS OF COLORED LINKS

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TESTUO DEGUCHI
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TOMOTADA OHTSUKI
Department of Mathematics, Faculty of Science,
University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, Japan

Let us discuss connection of the new colored link invariants to the multivariable Alexander polynomial. It was shown by J. Murakami [15] that a colored link invariant which corresponds to $\hat{\Phi}(L, \alpha)$ for the N=2 case is a version of the multivariable Alexander polynomial (the Conway potential function). Therefore the new colored link invariants $\hat{\Phi}(L, \alpha)$ for $N = 3, 4, \ldots,$ are generalizations of the multivariable Alexander polynomial.

$$x + 3 + x^{-1}$$

$$x^2 + 3x + 5 + 3x^{-1} + x^{-2}$$

$$x^3 + 3x^2 + 6x + 7 + 6x^{-1} + 3x^{-2} + x^{-3}$$

Osaka J. Math.
45 (2008), 541–564

COLORED ALEXANDER INVARIANTS AND CONE-MANIFOLDS

JUN MURAKAMI

1. Introduction

New link invariants are introduced in [1] for colored links. They are defined for each positive integer N and considered as a generalization of the multivariable Alexander polynomial [12], which corresponds to the case $N = 2$. Here we redefine these invariants by using the universal R-matrix of $\mathcal{U}_q(sl_2)$.

Let $q = \exp(\pi\sqrt{-1}/N)$ be a $2N$-th root of unity. Let $\mathcal{U}_q(sl_2)$ be the quantum enveloping algebra corresponding to the Lie algebra sl_2 defined by the following genera-

$$R^{ij}_{kl} = q^{\frac{1}{2}(\lambda-2i-2n)(\mu-2j+2n)+n(n-1)/2}\,\frac{\{i+n;n\}\{\mu-j+n;n\}}{\{n;n\}}$$

Osaka J. Math.
45 (2008), 541–564

COLORED ALEXANDER INVARIANTS AND CONE-MANIFOLDS

JUN MURAKAMI

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1990-2020: connection to physics, QFT, ... ???

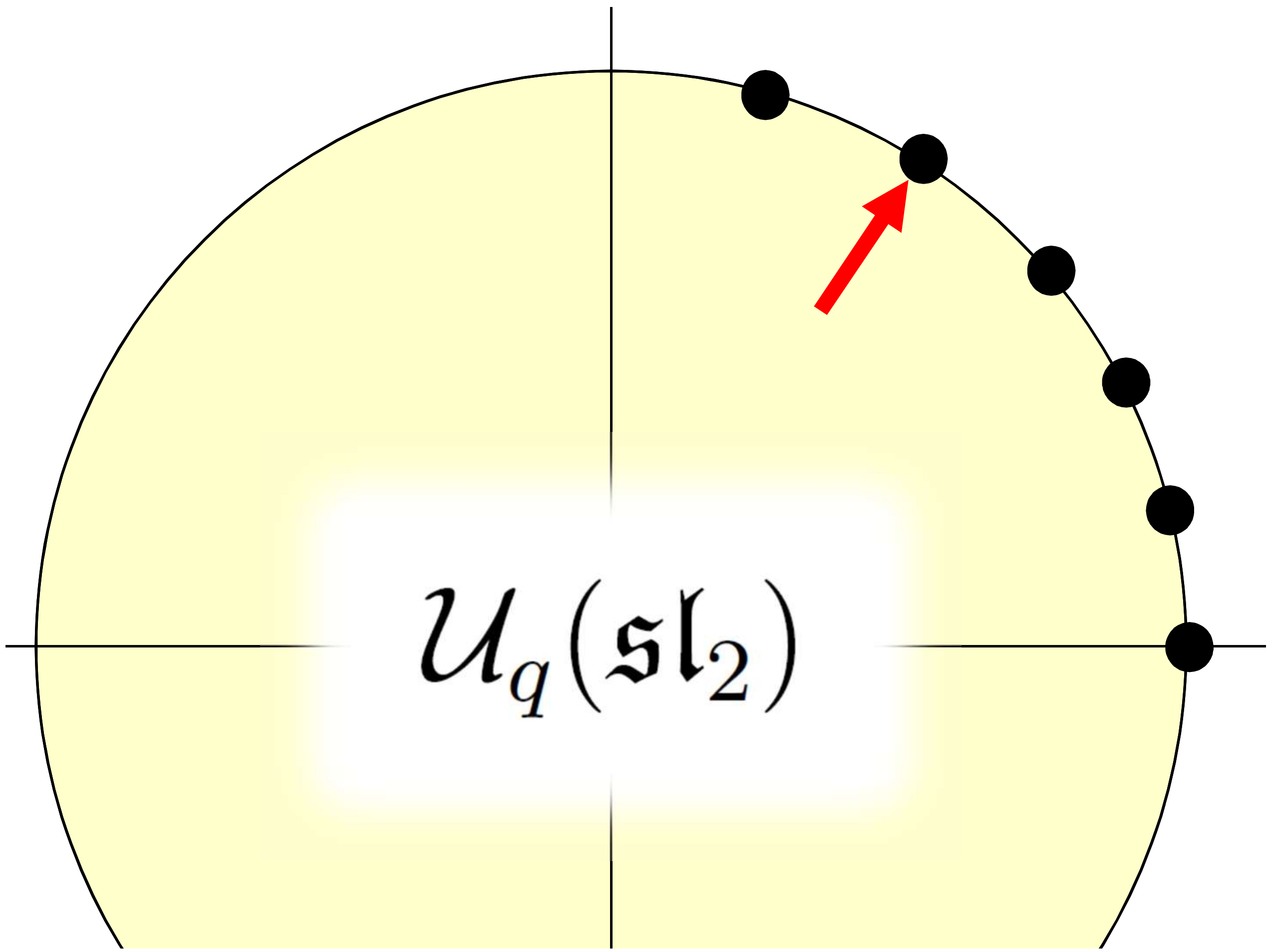
quantum groups
at generic q

non-semisimple
logarithmic

$$R_{\text{Park}}(x, y; q) \xrightarrow{q \to \text{root of } 1} R_{\text{Murakami}}(x, y; q)$$

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko
F.Costantino, S.G., P.Putrov

New explicit predictions for higher-rank analogues of ADO / CGP invariants



$$KEK^{-1} = q^2 E\,, \qquad KFK^{-1} = q^{-2} F\,, \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$\overline{\mathcal{U}}_q(\mathfrak{sl}_2)$$

$$KEK^{-1} = q^2 E\,, \quad KFK^{-1} = q^{-2} F$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$q = e^{\frac{i\pi}{p}}$$

2p-th root of unity

"restricted quantum group"

$$E^p = 0 = F^p \qquad , \qquad K^{2p} = \mathbf{1}$$

$$\overline{\mathcal{U}}_q(\mathfrak{sl}_2) \qquad KEK^{-1} = q^2 E\,, \quad KFK^{-1} = q^{-2} F$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$q = e^{\frac{i\pi}{p}}$$

2p-th root of unity

"restricted quantum group"

$$E^p = 0 = F^p \quad , \quad K^{2p} = \mathbf{1}$$

$$\Rightarrow \dim \overline{\mathcal{U}}_q(\mathfrak{sl}_2) = 2p^3$$

$\overline{\mathcal{U}}_q(\mathfrak{sl}_2)$

2p-th root of unity
"restricted quantum group"

$$\dim \overline{\mathcal{U}}_q(\mathfrak{sl}_2) = 2p^3$$

$$\dim \mathfrak{Z} \;=\; 3p-1$$

SL(2,ℤ)

S and T matrices of
a log-VOA

$\mathcal{U}_q(\mathfrak{sl}_2)$ $q = e^{\frac{i\pi}{p}}$ → *root of 1*

(3p-1)-dimn'l
SL(2,$\mathbb{Z}$) rep

→ *semisimpl.*

(p-1)-dimn'l
SL(2,$\mathbb{Z}$) rep

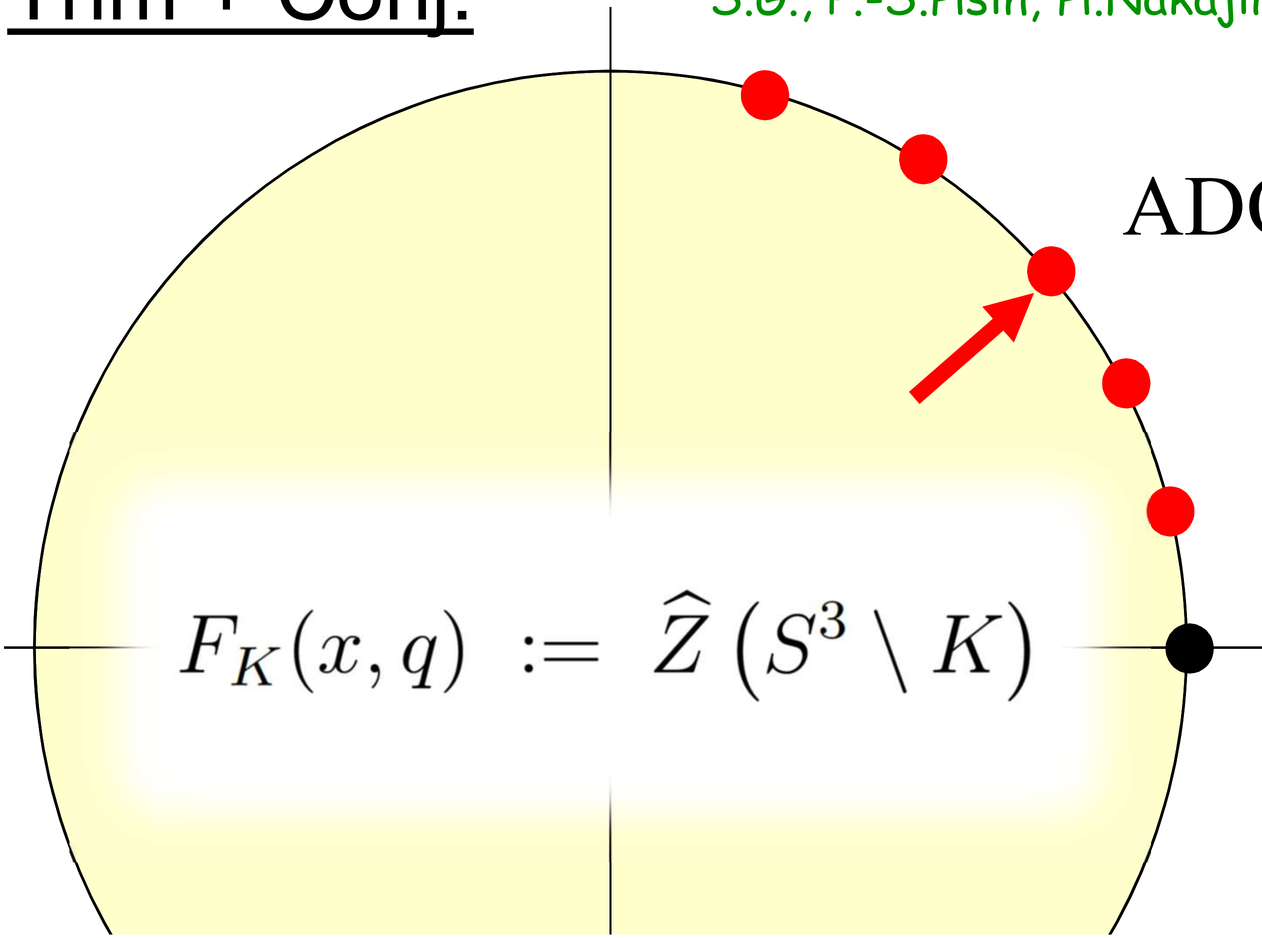
Triplet (1,p) model

WZW $SU(2)_{p-2}$

B.Feigin, A.Gainutdinov, A.Semikhatov, I.Tipunin
T.Creutzig, A.Gainutdinov, I.Runkel
C.Negron
P.Etingof, V.Ostrik
M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison
T.Creutzig, T.Dimofte, N.Garner, N.Geer
B.Feigin, S.G., N.Reshetikhin
:

Thm + Conj:

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko



$$F_K(x,q) \; := \; \widehat{Z}\left(S^3 \setminus K\right)$$

ADO polynomials

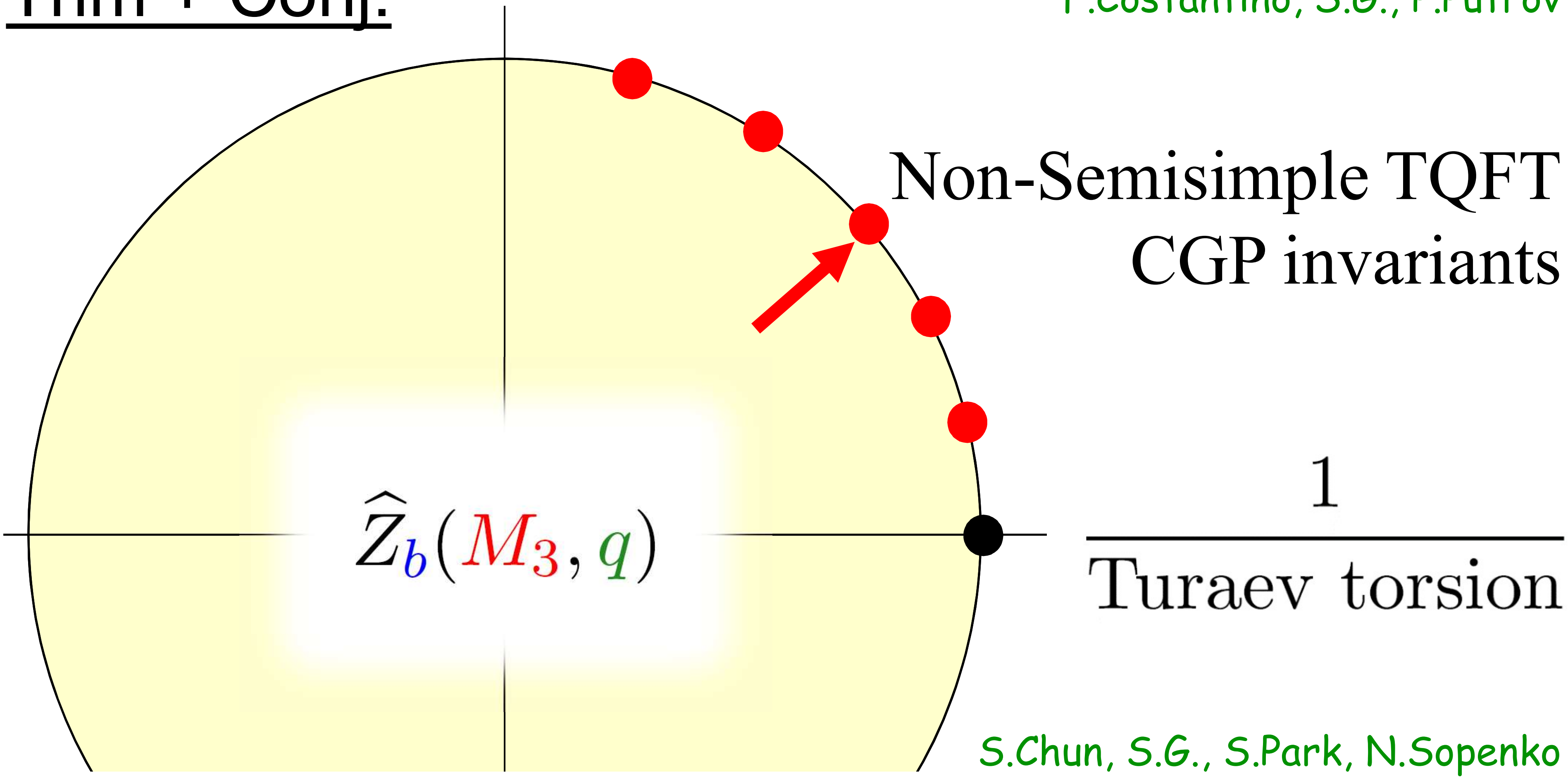
$$\frac{1}{\text{Alexander}}$$

S.G., C.Manolescu

$$F_K(x,q)|_{q=\zeta_p} \; = \; \frac{\mathrm{ADO}_p(x/\zeta_p; K)}{\Delta_K(x^p)}, \qquad \zeta_p = e^{2\pi i/p}$$

Thm + Conj:

F.Costantino, S.G., P.Putrov

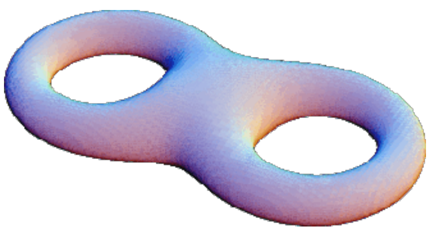


S.Chun, S.G., S.Park, N.Sopenko

$$N_r(M_3, \omega) = \sum_b c^{\mathrm{CGP}}_{\omega,b} \widehat{Z}_b(M_3, q)\Big|_{q \to e^{\frac{2\pi i}{r}}}$$

F.Costantino, S.G., P.Putrov

	CGP	GPPV
symmetry G	$U(1)$ 0-form	$\mathbb{Z}$ 1-form
charged objects	$\mathbb{Z}$ 0-dimensional	$U(1)$ 1-dimensional
$V(\Sigma)$ decorated by	$H^1(\Sigma, U(1))$	$H^2(\Sigma, \mathbb{Z})$
$V(\Sigma)$ graded by	$H^2(\Sigma, \mathbb{Z})$	$H^1(\Sigma, U(1))$



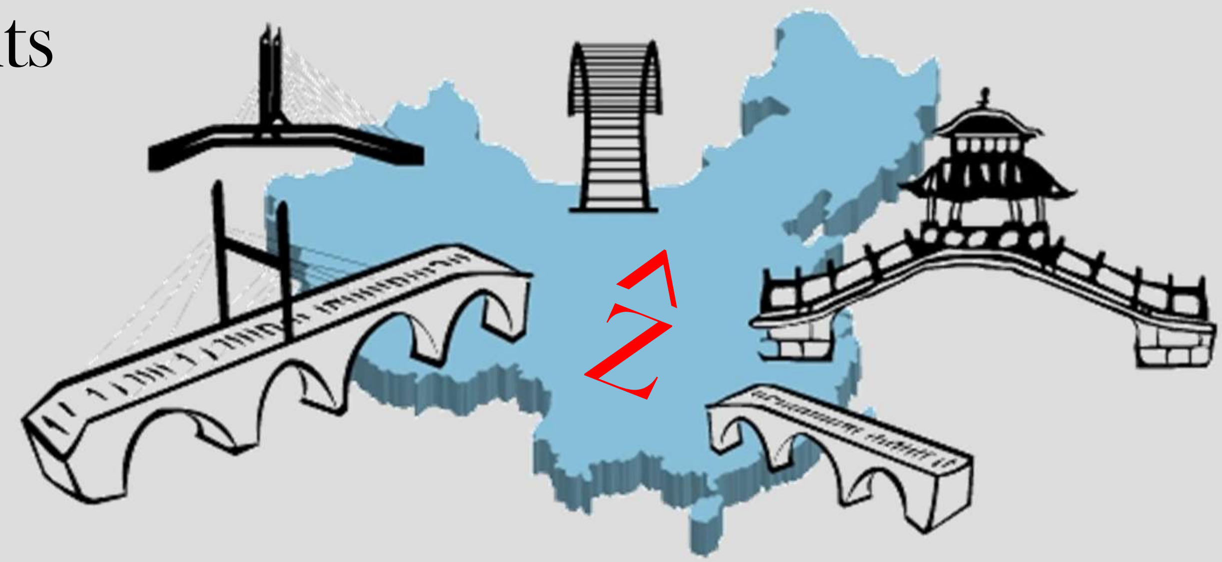
C.Blanchet, F.Costantino, N.Geer, B.Patureau-Mirand
M.Barkeshli, P.Bonderson, M.Cheng, Z.Wang
T.Khandhawit, J.Lin, H.Sasahira
A.Juhasz, I.Zemke
⋮

- Fourier transform of a decorated TQFT
- r-wrapping of a graded TQFT

ADO/CGP
invariants

cobordism
invariants

Turaev
torsion



WRT

Andersen-Kashaev
TQFT

equivariant Verlinde
formula

Conj, then Thm: $\widehat{A}^{(p)}\,\mathrm{ADO}_p(x) \;=\; 0$

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko
J.Brown, T.Dimofte, S.Garoufalidis, N.Geer

follows from $\widehat{A}\,Z\left(S^3 \setminus K\right) = 0$

$$\mathrm{WRT}(M_3,k) \;=\; \sum_b c_b^{\mathrm{WRT}}\;\widehat{Z}_b(q)\Big|_{q\to e^{\frac{2\pi i}{k}}}$$

Relation to other 3-manifold invariants labeled by Spin or Spin^c structures?


R.Kirby, P.Melvin

$$\sum_{s\in\mathrm{Spin}(M_3)} \exp\left(-2\pi i\frac{3\mu(M_3,s)}{16}\right) = \mathrm{WRT}\,(M_3,k)\,\Big|_{k=4}$$

$\underline{q \to 0:}$ $\qquad \widehat{Z}_b(M_3; q) \sim q^{\Delta_b}$ $\qquad$ S.G., S.Park, P.Putrov

$b \in \mathrm{Spin}^c(M_3)$

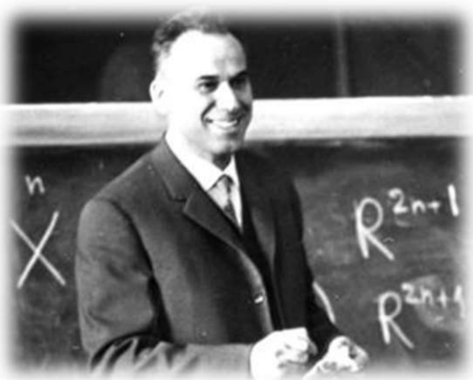
related to correction terms

$$d(M_3, b) \in \mathbb{Q}$$

new proofs of Donaldson's diagonalization theorem and the Thom Conjecture

P.Ozsvath, Z.Szabo

$\underline{q \to i:}$

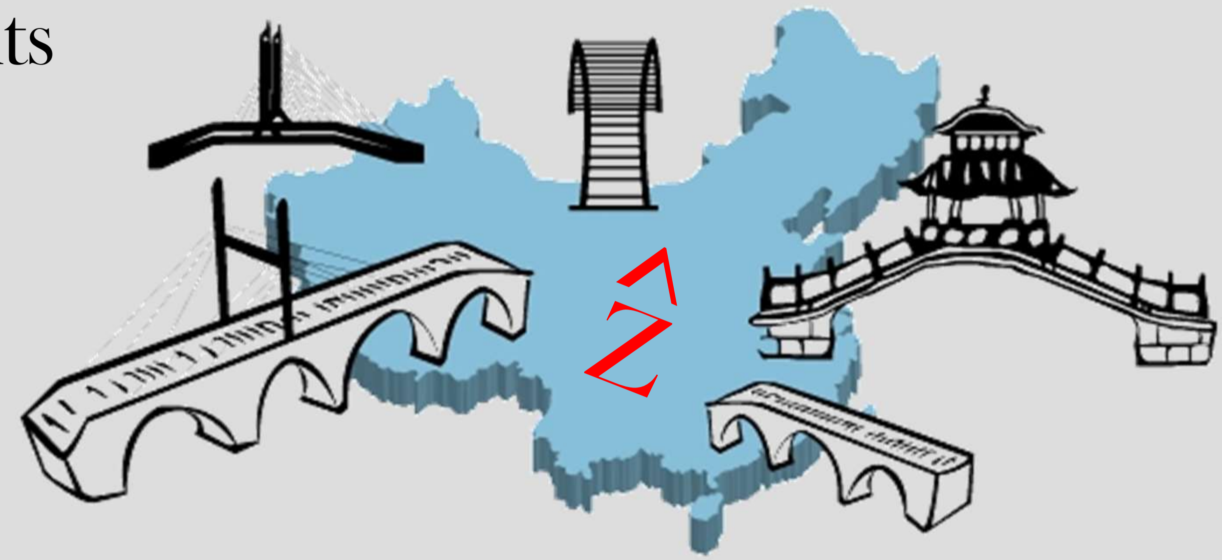


$$\exp\left(-2\pi i \frac{3\mu(M_3, s)}{16}\right) = \sum_b c_{s,b}^{\mathrm{Rokhlin}} \; \widehat{Z}_b(M_3, q)\Big|_{q=i}$$

ADO/CGP
invariants

cobordism
invariants

Turaev
torsion



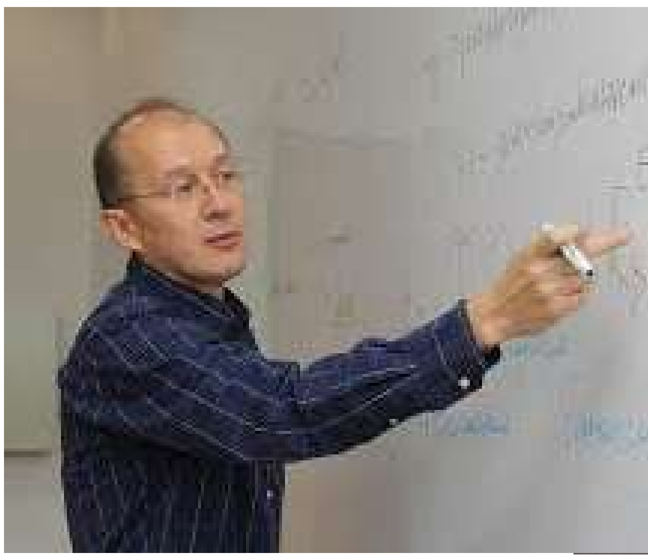
WRT

Andersen-Kashaev
TQFT

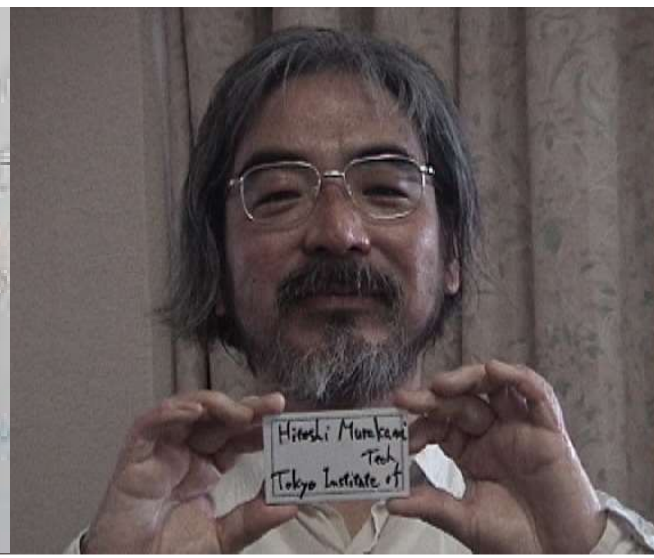
equivariant Verlinde
formula

Back in 1990s ...

$$\lim_{n\to\infty} \frac{2\pi \log |J_n(K; q = e^{2\pi i/n})|}{n} = \mathrm{Vol}\big(S^3 \setminus K\big)$$



Rinat Kashaev



Hitoshi Murakami

Jun Murakami

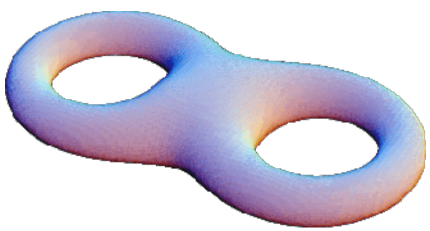
Communications in
Mathematical Physics

Three-Dimensional Quantum Gravity, Chern-Simons Theory, and the A-Polynomial

Sergei Gukov

Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA

A-polynomial = spectral curve (analogue of the Seiberg-Witten curve for 3d $\mathcal{N}=2$ theory)

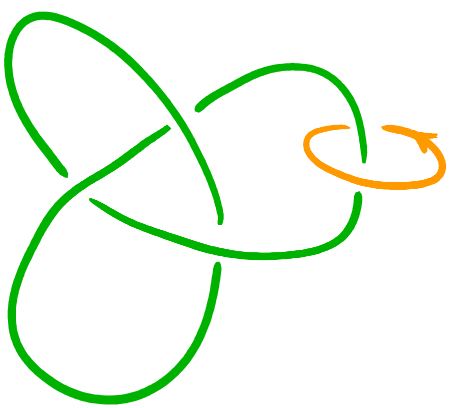


leads to many generalizations ...

$$q = e^{\hbar} \to 1\,, \qquad n \to \infty\,, \qquad q^n = x \quad \text{(fixed)}$$

Exact Results for Perturbative Chern-Simons Theory with Complex Gauge Group

Tudor Dimofte,[1] Sergei Gukov,[1,2] Jonatan Lenells,[3] and Don Zagier[4,5]

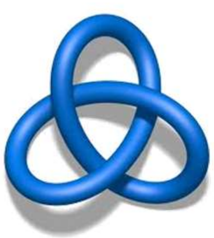


$$\mathrm{Hol} = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}$$

$$Z_{\mathrm{pert}}^{(\alpha)}(S^3 \setminus K; \hbar, x) \; = \; \exp\left(\frac{1}{\hbar} S_0^{(\alpha)}(x) + S_1^{(\alpha)}(x) + \hbar S_2^{(\alpha)}(x) + \ldots \right)$$

$$
\begin{aligned}
Z^{(\mathrm{ab})}_{\mathrm{pert}}(\mathbf{3_1}) &= (x^{1/2} - x^{-1/2} - x^{5/2} + x^{-5/2} - x^{7/2} + x^{-7/2} + \ldots) \\
&+ \hbar(x^{1/2} - x^{-1/2} - 2x^{5/2} + 2x^{-5/2} - 3x^{7/2} + 3x^{-7/2} + \ldots) \\
&+ \frac{\hbar^2}{2}(x^{1/2} - x^{-1/2} - 4x^{5/2} + 4x^{-5/2} - 9x^{7/2} + 9x^{-7/2} + \ldots) \\
&+ \frac{\hbar^3}{6}(x^{1/2} - x^{-1/2} - 8x^{5/2} + 8x^{-5/2} - 27x^{7/2} + 27x^{-7/2} + \ldots) \\
&+ \frac{\hbar^4}{24}(x^{1/2} - x^{-1/2} - 16x^{5/2} + 16x^{-5/2} - 81x^{7/2} + 81x^{-7/2} + \ldots) \\
&+ \frac{\hbar^5}{120}(x^{1/2} - x^{-1/2} - 32x^{5/2} + 32x^{-5/2} - 243x^{7/2} + 243x^{-7/2} + \ldots) \\
&+ \ldots \\
&= \sum_{m\geq 1} f_m(q) \cdot \left(x^{\frac{m}{2}} - x^{-\frac{m}{2}}\right) = Z_{\mathrm{vortex}}(q;x)
\end{aligned}
$$

can be independently computed from 3d $\mathcal{N}=2$ theory or open topological string (open BPS invariants)



$$f_1 = -q, \quad f_3 = 0, \quad f_5 = q^2, \quad f_7 = q^3, \quad f_9 = 0, \quad \ldots$$

singularities on the Borel plane: $\ell_{\alpha\beta} = S_\beta - S_\alpha$

$$\mathcal{S}_\theta Z_\alpha^{\text{pert}}(\hbar) = Z_\alpha^{\text{pert}}(\hbar) + \sum_\beta n_\beta^\alpha e^{-\frac{\ell_{\alpha\beta}}{\hbar}} Z_\beta^{\text{pert}}(\hbar)$$

S.G., M.Marino, P.Putrov
J.Andersen, W.Mistegaard

Theorem: $n_\beta^\alpha = 0$ $\quad \alpha = \text{irreducible}, \quad \beta = \text{abelian}$

Corollary:
$$Z_{\text{CS}}(M_3) = \sum_{a \in \text{abelian}} e^{2\pi i k S_a} Z_a(M_3)$$

$$Z_a^{\text{pert}}(\hbar) \xrightarrow[\text{(Borel sum)}]{\text{resurgence}} Z_a(q = e^{\hbar}) \xrightarrow{S_{ab}} \widehat{Z}_b(q)$$

Conj: modular group acts on

S.G., P.Putrov, C.Vafa
M.Abouzaid, C.Manolescu
S.Meinhardt
:

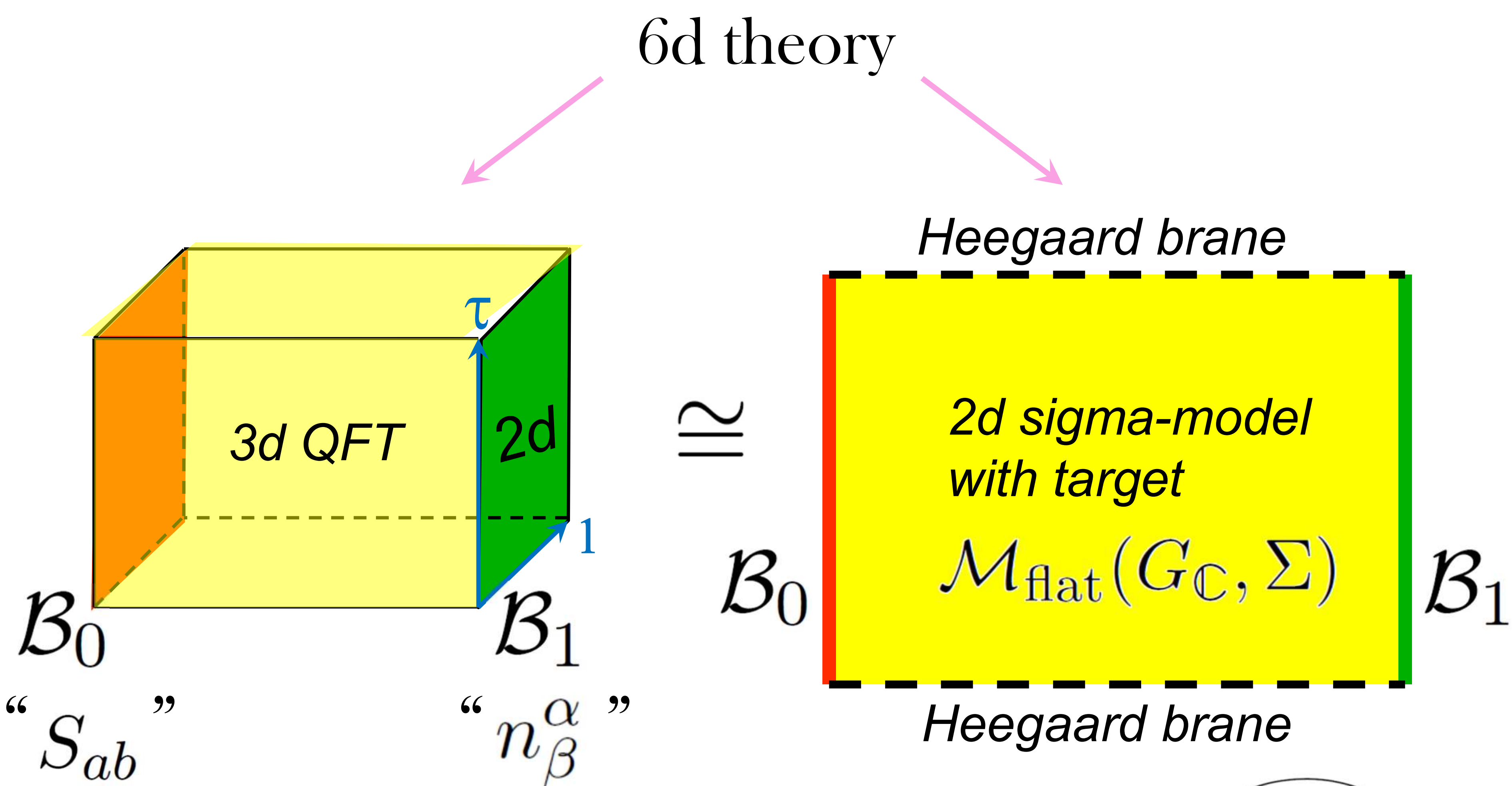
$$\mathrm{HF}_{G_{\mathbb{C}}}(M_3) = K^0(\mathrm{MTC}[M_3, G])$$

$$= \mathrm{HP}(M_3) + \text{(reducibles)} \overset{\otimes\,\mathbb{C}(q)}{\sim} \mathrm{Sk}(M_3)$$

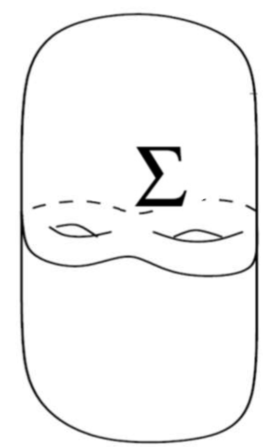
cf. B.Feigin, S.G.
M.Dedushenko, S.G., H.Nakajima, D.Pei, K.Ye
S.Gunningham, D.Jordan, P.Safronov
:

e.g. Poincaré sphere:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{3\pi i/8} \end{pmatrix} \qquad S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$



boundary condition (brane) $\mathcal{B}_1$
depends on Spin^c structure
or cohomology class ω

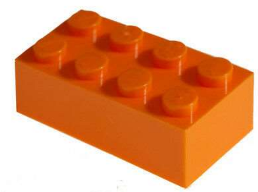
$M_3 =$ 

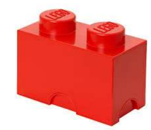
➡ **Conj:**

$$\text{Andersen-Kashaev invariants} = \sum_{a,b} c^{\mathrm{AK}}_{a,b}\, \widehat{Z}_a(q)\widehat{Z}_b(\widetilde{q})$$

$$\text{equivariant Verlinde formula} = \sum_{a,b} c^{\mathrm{eq.Ver.}}_{a,b}\, \widehat{Z}_a(q,t)\widehat{Z}_b(q^{-1},t)\Big|_{q\to e^{\frac{2\pi i}{k}}}$$

“ $n^\alpha_\beta + S_{ab}$ ”





cf.

$$N_r(M_3,\omega) = \sum_b c^{\mathrm{CGP}}_{\omega,b}\, \widehat{Z}_b(M_3,q)\Big|_{q\to e^{\frac{2\pi i}{r}}}$$

$$\mathrm{WRT}(M_3,k) = \sum_b c^{\mathrm{WRT}}_b\, \widehat{Z}_b(q)\Big|_{q\to e^{\frac{2\pi i}{k}}}$$

$$\exp\left(-2\pi i \frac{3\mu(M_3,s)}{16}\right) = \sum_b c^{\mathrm{Rokhlin}}_{s,b}\, \widehat{Z}_b(M_3,q)\Big|_{q=i}$$


ADO/CGP
invariants
Turaev
torsion
cobordism
invariants
$\hat{Z}$
WRT
Andersen-Kashaev
TQFT
equivariant Verlinde
formula