Large charge asymptotic expansion for superconformal indices

Work with Ji Hoon Lee

Giant graviton expansion

- Applies to superconformal indices of large N gauge theories
  - Holographic interpretation.
- Also applies to index-like generating functions
  - General combinatorial interpretation?
  - Property of non-perturbative string theory?
- Categorification?

Basic statement I

$$Z_N(x;y_*) = \sum_{n,m_i} c^{(N)}_{n;m_i} x^n \prod_i y_i^{m_i} = \sum_{m_i} c^{(N)}_{m_i}(x) \prod_i y_i^{m_i}$$

- Formal power series, but may actually converge for |x|<1, |y|<1.
- Large N, finite n,m: coefficients stabilize: $Z_\infty(x;y_*) = \sum_{n,m_i} c^{(\infty)}_{n;m_i} x^n \prod_i y_i^{m_i}$
- Corrections at order N. Focus on finite m, large n:

$$\frac{Z_N(x;y_*)}{Z_\infty(x;y_*)} = 1 + x^N \sum_{n,m_i} \hat{d}^{(1)}_{n;m_i} x^n \prod_i y_i^{m_i} + O(x^{2N})$$

Basic statement II

- N dependence only in overall exponential.
- Infinite series of corrections $$\frac{Z_N(x;y_*)}{Z_\infty(x;y_*)} = 1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x;y_i)$$
- Coefficients determined from large N behaviour, but formula holds for all N
    - Even N<0, gives 0
    - Arbitrarily large charges, but requires high k

Back to example

- 1/2 BPS operators/states for U(N) SYM
    - U(N)-invariant polynomials in adjoint X
    - Traces $\mathrm{Tr} X^n$ up to trace relations: $Z_N(x) = \frac{1}{\prod_{n=1}^{N}(1-x^n)}$

$$\frac{Z_N(x)}{Z_\infty(x)} = \prod_{n=1}^{\infty}(1-x^{N+n}) = 1 - x^{N+1} - x^{N+2} - x^{N+3} - x^{N+4} - x^{N+5} - x^{N+6} \cdots$$

$$\frac{Z_N(x)}{Z_\infty(x)} = 1 - x^N \frac{x}{1-x} + x^{2N+3} + x^{2N+4} + 2x^{2N+5} + 2x^{2N+6} + 3x^{2N+7} + 3x^{2N+8} \cdots$$

$$\frac{Z_N(x)}{Z_\infty(x)} = 1 - x^N \frac{x}{1-x} + x^{2N} \frac{x^3}{(1-x)(1-x^2)} - x^{3N+6} - x^{3N+7} - 2x^{3N+8} - 3x^{3N+8} \cdots$$

- $$\frac{Z_N(x)}{Z_\infty(x)} = 1 + x^N \frac{1}{1-x^{-1}} + x^{2N} \frac{1}{(1-x^{-1})(1-x^{-2})} + \cdots$$

Basic statement III

- Resum x —> 1/x to define map $Z_N(x;y_i) \to \tilde{Z}_k(x^{-1};y_i)$

$$\hat{Z}_k(x;y) = \sum d_m^{(k)}(x) \prod_i y_i^{m_i} \qquad \tilde{Z}_k(x^{-1};y) = \sum d_m^{(k)}(x) \prod_i y_i^{m_i}$$

- Observations
    - Example: $\tilde{Z}_k(x^{-1}) = Z_k(x^{-1})$
    - U(N) SYM: $Z_N(x;y,z,p,q) \leftrightarrow Z_k(x^{-1};p,q,y,z)$
    - Good large k limit $\tilde{Z}_\infty(x^{-1};y_i)$, inversion formula $\tilde{\tilde{Z}}_N(x;y_i) = Z_N(x;y_i)$
    - M2 <——> M5

Back to example II

- 1/2 BPS operators/states for U(N) SYM
  - U(N)-invariant polynomials in adjoint X
  - Traces $\mathrm{Tr} X^n$ up to trace relations: $Z_N(x) = \frac{1}{\prod_{n=1}^{N}(1-x^n)}$
- Holography: finite n traces <—> gravitons in AdS5xS5
  - n~N D-branes appear: giant gravitons
  - k giant gravitons support U(k) SYM, with x <-> 1/x
- Finite N = Gravitons + analytic continuation of giant gravitons?

Back to example III

- Giant gravitons = determinants (det X)^k
    - Fluctuations of giant gravitons: replace X's with oder symbols (or 1)
    - Emergent U(k) gauge theory $\det_{kN}(X \otimes \delta^a_b + \epsilon O^a_b)$
    - We will count these, coincide with $\tilde{Z}_k$
- Finite N = Gravitons + analytic continuation of giant gravitons?

4d SUSY gauge theories

- No solitons: polynomials in BPS (derivatives of) fields
  - E.g chiral multiplet: $\partial_{z_1}^n \partial_{z_2}^m X \to x p^n q^m$ $\qquad \partial_{z_1}^n \partial_{z_2}^m \psi_X \to -x^{-1} p^{n+1} q^{m+1}$
- Count individual "letters": $f(x,p,q) = \dfrac{x - x^{-1}pq}{(1-p)(1-q)}$
- Counts polynomials in the letters $\quad f \to \mathrm{PE}[f] = \prod_{n,m} \dfrac{(1 - x^{-1}p^{n+1}q^{n+1})}{(1 - xp^n q^n)}$
- Project onto gauge invariant polynomials

4d U(N) adjoint gauge theories

- Gauge-invariant operators built from adjoint "letters"
  - Count letters $f = f_{\mathrm{Adj}}(z)$
  - Apply formula $Z_N(z) = \frac{1}{N!}\int \frac{d\mu_a}{2\pi i \mu_a}\left[\prod_{a\neq b}(1-\mu_a/\mu_b)\right]\mathrm{PE}\left[(\sum_a \mu_a)(\sum_b \mu_b^{-1})f(z)\right]$
- Large N at fixed charge: count single-trace operators i.e. cyclic words
  - $Z_\infty = \frac{1}{\prod_{n=1}^{\infty}(1-f(z^n))}$ exact for less than N+1 letters

Near half-BPS expansion

$$\frac{Z_N(x;y)}{Z_\infty(x;y)} = \sum_k x^{kN} \hat{Z}_k(x;y)$$

- Analytic continuation $\tilde{x} = x^{-1}$ $\hat{Z}_k(x;y) \to \tilde{Z}_k(\tilde{x};y)$
- (True or fake) index of a "dual" U(k) gauge theory $\tilde{Z}_k(\tilde{x};y) = Z_k[\tilde{f}]$
- Explicit formula: $\frac{1-\tilde{f}}{1-\tilde{x}} = \frac{1-x}{1-f}$ $1-f = \frac{(1-x)(1-y)(1-z)}{(1-p)(1-q)}$
- E.g. N=4 SYM $\tilde{f}(\tilde{x},y,z,p,q) = f(\tilde{x},p,q,y,z)$
- Extension to (anti)fundamental matter $\tilde{f}_v = -\tilde{x}^{\frac{1}{2}} \frac{1-\tilde{f}}{1-\tilde{x}} f_v$,quivers, etc.

$1-f = (1-x)(1-h(y))$ $f = x + h(y) - xh(y)$

Fermionization trick

- Add auxiliary (anti)fundamental fermions $\det X = \int d\psi d\chi e^{\psi X \chi}$
- Open string insertions: X->Y inserts $\psi Y \chi$
- Add antifields for fermions for Ward identities
- Determinant modifications “counted” by $\tilde{Z}_k$

Questions

- Categorification?
  - Determinant analysis works best in complex Q O’ = [X,O]
  - General statement about Q-cohomology?
- Is it combinatorial?
- Multi-fugacity expansion? Wallcrossing?
- Black holes?