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Graduiertenkolleg 2149
Research Training Group

CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

Noncommutative QFT meets Blobbed Topological Recursion

New paths towards exact solutions of QFT's.

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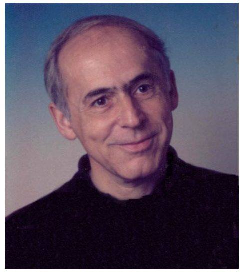

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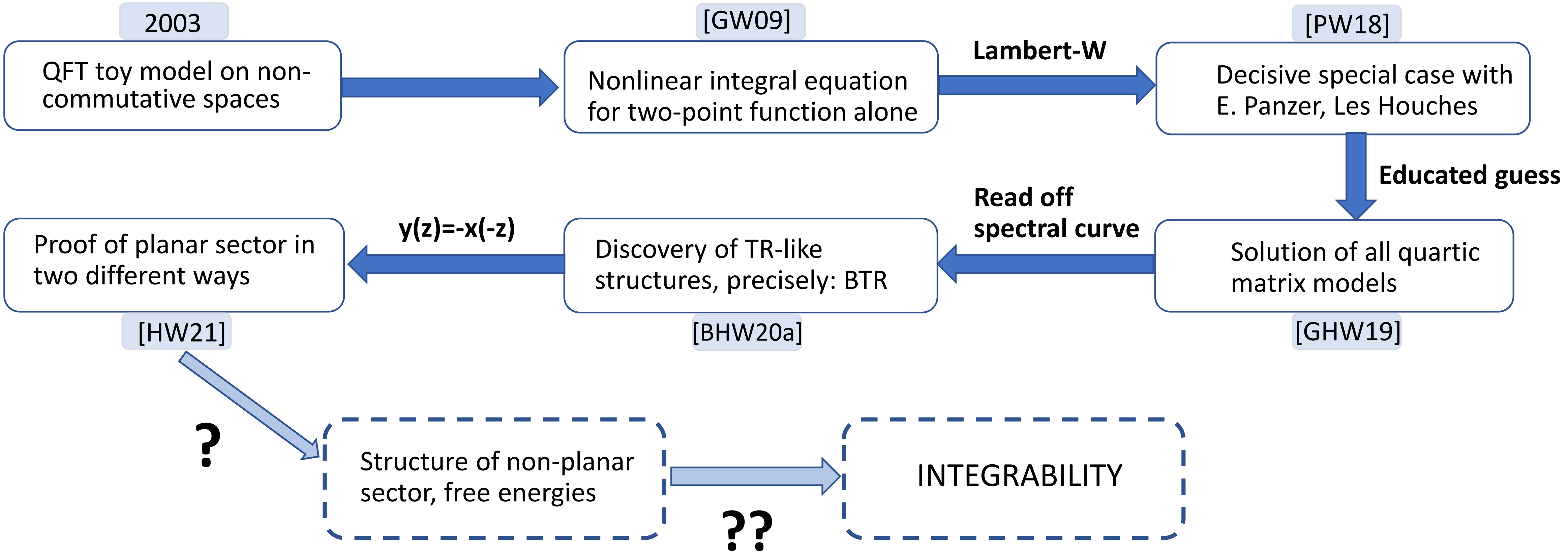
The path of the Grosse-Wulkenhaar model towards topological recursion:



- *[GW09]: Progress in solving a noncommutative quantum field theors in four dimensions.*
- *[PW18]: Lambert-W solves the noncommutative Φ4-model*
- *[GHW19]: Solution of all quartic matrix models*
- *[BHW20a], [HW21]: Blobbed topological recursion of the quartic Kontsevich model, Part I/II.*

QFT on the Moyal space

- Suppose one day **quantum theory** and **general relativity** will be unified: a suitable space will be needed – a noncommutative one.
- The **Moyal space** extends our picture of a product of quantum fields:

Remark: All that is about **Euclidean** QFT, introducing **time** (Wick rotation etc.) is far more complicated!

$$(g \star h)(x) = \int_{\mathbb{R}^D} \frac{dk}{(2\pi)^D} \int_{\mathbb{R}^D} dy\, g(x + \tfrac{1}{2}\Theta k)\, h(x+y) e^{ik\cdot y},$$

$$\Theta = \mathrm{id}_{D/2} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot 4V^{2/D}, \quad V \in \mathbb{R}, \quad x \in \mathbb{R}^D, \quad g,h \in \mathcal{S}(\mathbb{R}^D)$$

- As a vector space equipped with the star-product, the **Moyal algebra** possesses a matrix base

$$(f_{nm} \star f_{kl})(x) = \delta_{m,k} f_{nl}(x), \qquad \int_{\mathbb{R}^D} dx\, f_{nm}(x) = 8\pi V \delta_{n,m}$$

- Expand quantum fields in that base: $\phi(x) = \sum_{n,m} \phi_{nm} f_{nm}(x) \qquad \phi \in \mathcal{S}(\mathbb{R}^D)$

A simple toy model: the action of the NC real QFT with quartic interaction of scalars:

$$S[\phi] := \frac{1}{8\pi}\int_{\mathbb{R}^D} dx \left(\frac{1}{2}\phi\left(-\Delta + \Omega^2\|2\Theta^{-1}x\|^2 + \mu^2\right)\star\phi + \frac{\lambda}{4}\phi^{\star,4}\right)(x)$$

Laplacian *oscillator term* *mass* *coupling constant*

1) Expand quartic term in matrix base:

$$\int_{\mathbb{R}^D} dx(\phi\star\phi\star\phi\star\phi)(x) = \sqrt{\det(2\pi\Theta)}\,\mathrm{Tr}(\phi^4)$$

2) Choose self-duality point $\Omega = 1$: kinetic term becomes matrix product as well!

$$S(\phi) = \sqrt{\det(\Theta/4)}\,\mathrm{Tr}\left(E\phi^2 + \lambda\frac{\phi^4}{4}\right)$$

For finite matrices: ***Quartic Kontsevich Model*** with partition function:

From the noncommutative **Moyal space** to a **matrix model**

Hermitian matrices *size* *ext. field*

$$\mathcal{Z} = \int_{H_N} d\phi \exp\left[-N\mathrm{Tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)\right]$$

According to Weyl, identify the (spectral) dimension of the QFT from the eigenvalue spectrum of the external field:

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Theorem (Weyl, 1911)

Given a spectral measure $d\varrho(t)$, uniquely determined by a Laplace(-Beltrami) operator *E*, a canonical dimension $D := \lfloor \mathfrak{D} \rfloor$ is provided by the spectral dimension

$$\mathfrak{D} = \inf\left\{ m \in \mathrm{R}_+ : \int \frac{d\varrho(t)}{(1+t)^{m/2}} < \infty \right\} \qquad D := \lfloor \mathfrak{D} \rfloor$$

Some examples of our model on the D-dim. Moyal space:

	D = 0	D = 2	D = 4
$d\varrho(t)$	▪ Discrete measure, truncated spectrum	▪ Trivial measure 1, continuous spectrum	▪ Linear measure, continuous spectrum
Multiplicity r_n *of eigenvalue* e_n *of E*	r_n arbitrary	$r_n = 1$	$r_n = n$

Correlation functions as derivatives of the partition function

- The most natural objects in QFT are the **correlation functions**

$$G^{(g)}_{|p_1^1 p_2^1 \dots p_{n_1}^1|\dots|\dots p_{n_b}^b|} N^{2-b-2g-(n_1+..+n_b)} := \langle \phi_{p_1^1 p_2^1} \phi_{p_2^1 p_3^1} .. \phi_{p_{n_1}^1 p_1^1} \phi_{p_1^2 p_2^2} .. \phi_{p_1^b p_2^b} .. \phi_{p_{n_b}^b p_1^b} \rangle_c$$

- The set of indices may be written as *b* disjoint cycles of length $l(\gamma_i) = n_i$
- By definition, their Feynman graph expansion generates only **fully simple** quadrangulations (in the sense of E. Garcia Failde '17), weighted/decorated due to external matrix
- **Starting point:** solution of the following non-linear integral equation (after a certain complexification, see later), the rest follows recursively:

$$\Big(\zeta + \eta + \frac{\lambda}{N} \sum_{k=1}^{d} r_k G^{(0)}(\zeta, e_k) + \frac{r_k}{e_k - \zeta}\Big) G^{(0)}(\zeta, \eta) - \frac{\lambda}{N} \sum_{k=1}^{d} r_k \frac{G^{(0)}(e_k, \eta)}{e_k - \zeta} = 1$$

Try to solve this equation for the 2-point function for any integral measure (d to infty), including mass and field renormalization: ***Obtain the solution of all quartic matrix models!***

Solution of all quartic matrix models

- The solution for any measure is encoded in a variable transform *R(z)*, solved by an integral equation:

$$R_D(z) = z - \lambda(-z)^{D/2} \int_{\nu_D}^{\Lambda_D^2} \frac{dt\, \varrho_\lambda(z)}{(t+\mu^2)^{D/2}(t+\mu^2+z)}$$

$$\nu_D := R_D^{-1}(0) \text{ and } \Lambda_D^2 := R_D^{-1}(\Lambda^2)$$

- The measure is deformed by the coupling constant. Using the Dirac distribution, in zero dimensions only a finite sum survives:

$$\varrho_\lambda(t) = \varrho_0(R_D(t)) \qquad \varrho_0(t) = \frac{1}{N}\sum_{k=1}^{N} r_k \delta\Big(t - \big(e_k - \frac{\mu_{bare}^2}{2}\big)\Big)$$

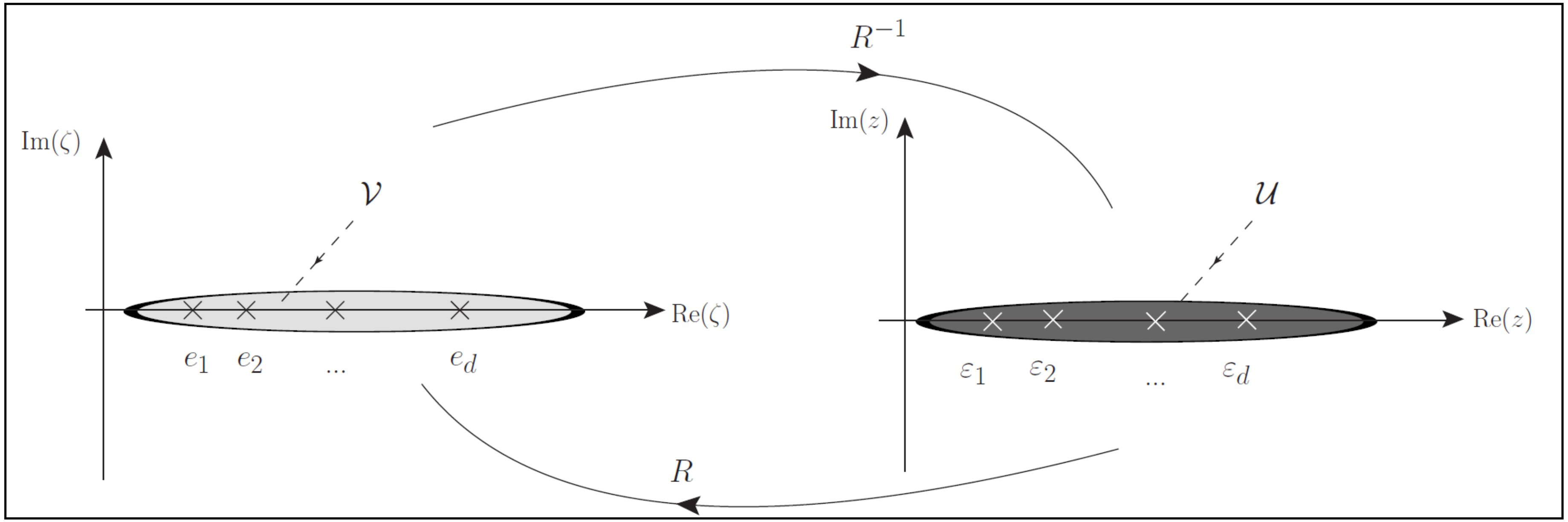
These formulae gives rise to a solution ***all quartic matrix models*** in terms of a (implicitly defined) function, with a characteristic deformation

Express the solution of the 2-point function via the transformation R(z). Here: finitely many eigenvalues of the external field, D = 0

Theorem [GHW19]

$$\mathcal{G}^{(0)}(z,w) = \frac{1 - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))} \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon_k}^j)}{R(w)-R(\varepsilon_j)}}{R(w)-R(-z)}$$

- *R(z)* transforms the eigenvalues in a simple manner:



Consider solutions of R(z) in D dimensions: R becomes constituent x(z) of the spectral curve!

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D = 0

- “truncated energy spectrum”, trivial for QFT, but most easy to investigate in BTR context
- Mainly investigated yet

$$x_{0D}(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{(\varepsilon_k + z)x'_{0D}(\varepsilon_k)}, \qquad x_{0D}(\varepsilon_k) = e_k$$

Send $d \to \infty$

D = 2

- Convenient curve, relation to Hurwitz numbers
- Fairly hard to prove – loop equations are complicated integral equations

$$x_{2D}(z) = z + \lambda \ln(1+z)$$

QUARTIC KONTSEVICH MODEL FOR DIMENSION {0,2,4}

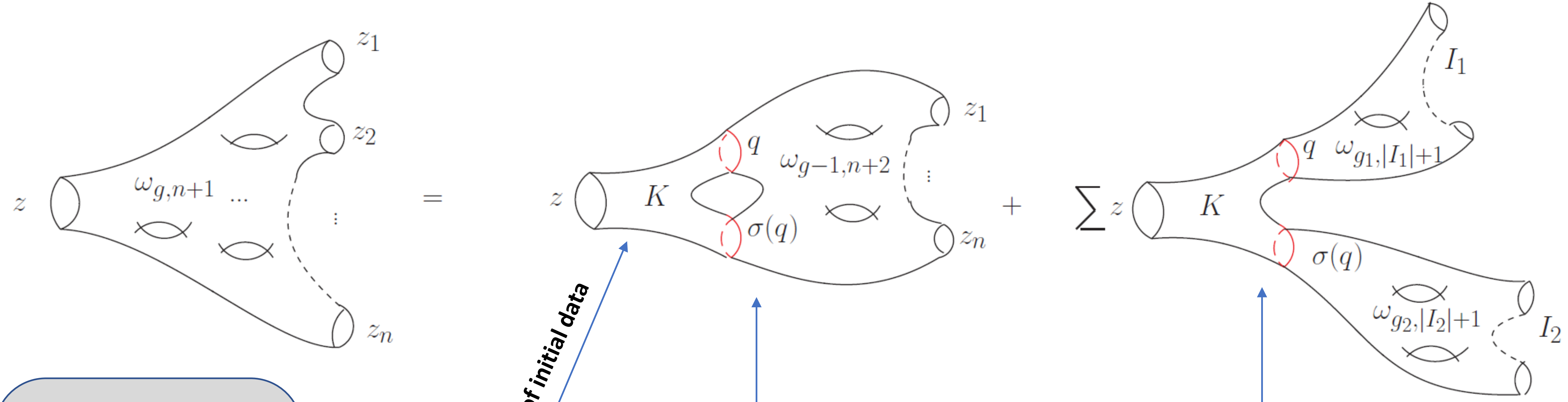
Send $d \to \infty$

D = 4

- Most interesting case for physicists, dimension corresponding to our world
- Even more complicated integral eq.
- Generating series of hyperlogarithms, guaranteeing non-triviality of the QFT
- Mass and field renormalization
- Still unclear if BTR occurs (non-algebraic ramification point!)

$$x_{4D}(z) = z\,{}_2F_1(\alpha_\lambda, 1-\alpha_\lambda, 2|-z) \qquad \alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$$

Recap: Start with initial data (unstable topologies) and compute the rest via recursion:



Kernel, made of initial data

„Glue the same object twice, create a hole“

„Glue all possible combinations of two objects“

$$\sum_{\beta_i}\operatorname*{Res}_{q\to\beta_i} K_i(z,q)\Big(\omega_{g-1,n+2}(I,q,\sigma_i(q)) + \sum_{\substack{g_1+g_2=g\\ I_1\uplus I_2=I\\ (g_1,I_1)\neq(0,\emptyset)\neq(g_2,I_2)}} \omega_{g_1,|I_1|+1}(I_1,q)\,\omega_{g_2,|I_2|+1}(I_2,\sigma_i(q))\Big)$$

Galois involution:

- Nearby ram. point:
  $x(z) = x(\sigma(z))$
- Fixed point:
  $\sigma(\beta_i) = \beta_i$
- not identity:
  $\sigma \neq \mathrm{id}$

Main achievement in 2020: the correlation functions themselves are a misleading path!

- Investigating the 2+2-point function it became clear that one has to keep the eyes on derivatives like:

$$T^{(g)}_{q_1,\dots,q_m\|p_1^1 p_2^1 \dots p_{n_1}^1|\dots|\dots p_{n_b}^b|} := \frac{(-N)^m}{r_{q_1}\dots r_{q_m}} \frac{\partial^m}{\partial e_{q_1}\partial e_{q_2}\dots\partial e_{q_m}} G^{(g)}_{|p_1^1 p_2^1 \dots p_{n_1}^1|\dots|\dots p_{n_b}^b|}$$

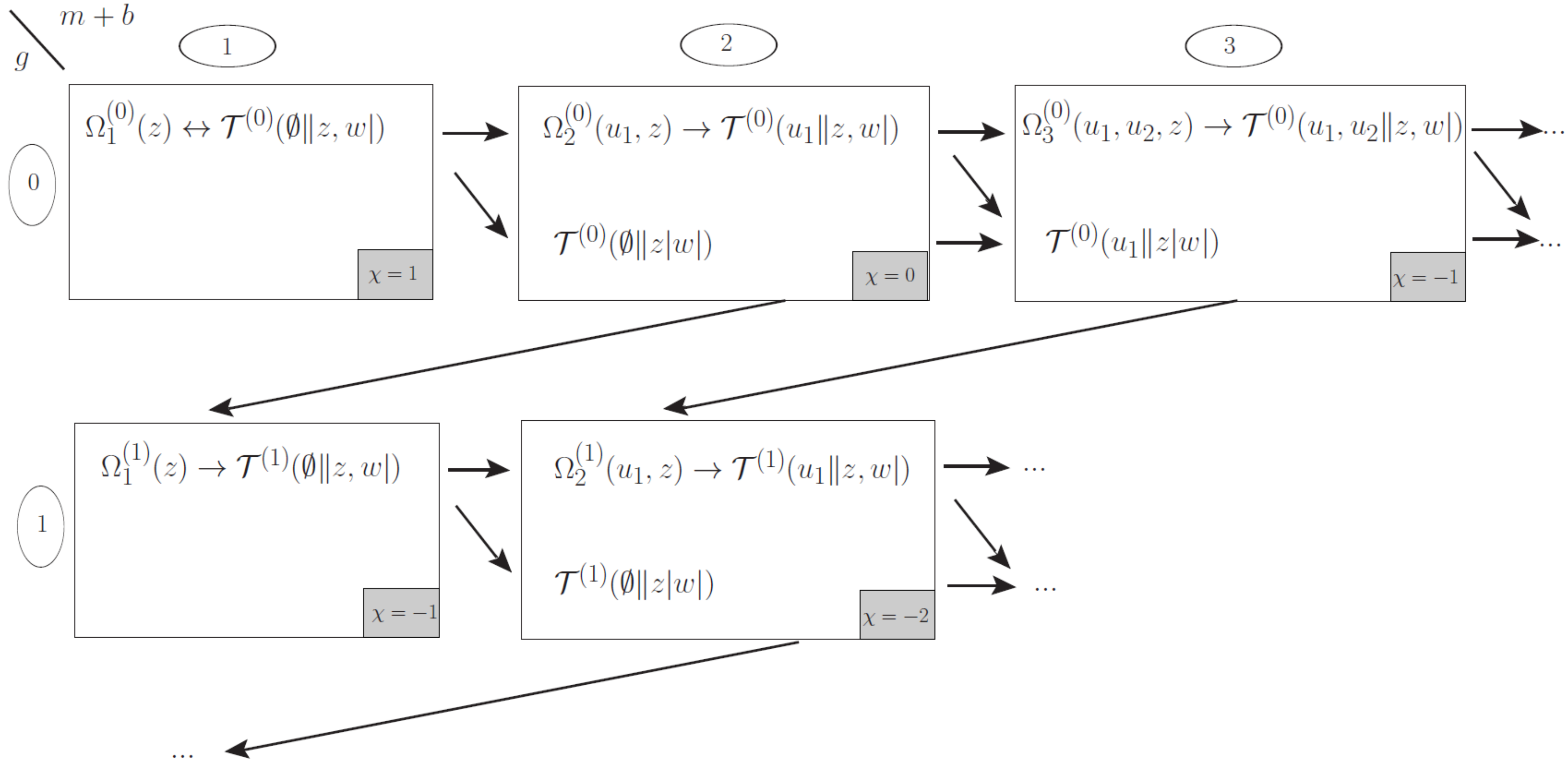
- This kind of derivative turns out to be the **boundary creation operator:** creates formally the objects obeying (B)TR from the free energies:

$$\Omega^{(g)}_{q_1,\dots,q_m} = \frac{(-N)^m}{r_{q_1}\dots r_{q_m}} \frac{\partial^m}{\partial e_{q_1}\partial e_{q_2}\dots\partial e_{q_m}} F^{(g)} + \frac{\delta_{m,2}\delta_{g,0}}{(e_{q_1}-e_{q_2})^2}$$

- Precisely, the objects later obeying (B)TR are obtained via the following easy transformation:

$$\omega_{g,n}(z_1,..,z_n) := \lambda^{2g+n-2}(\Omega_{g,n}(z_1,..,z_n) - \delta_{g,0}\delta_{n,1}V'(x(z)))dx(z_1)..dx(z_n)$$

The following triangular pattern gives the complete solution of the generalized Schwinger-Dyson equations



The unstable topologies: The reflection symmetry is the main phenomenon for the whole model!

! With x(z)=R(z) and symmetry, everything is constructed from R

disc amplitude

$$\omega_{0,1}(z_1) = y\,dx \quad \underline{y(z) = -x(-z)}$$

Global symmetry of the spectral curve affects all topologies: !

cylinder amplitude

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1\,dz_2}{(z_1 - z_2)^2} + \frac{dz_1\,dz_2}{(z_1 + z_2)^2}$$

Bergman kernel *reflected Bergman kernel*

- Only the standard bidifferential is taken into the recursion kernel, first hint for BTR

Leaving the original correlators of QFT behind gives rise to an astonishing simplification!

Proposition [BHW20b]

$$\Omega^{(g)}_{q_1,q_2} = \frac{\delta_{g,0}}{(E_{q_1}-E_{q_2})^2} + \sum_{g_1+g_2=g} G^{(g_1)}_{|q_1q_2|} G^{(g_2)}_{|q_1q_2|}$$
$$+ \frac{1}{N^2}\sum_{k,l=1}^{N} G^{(g)}_{|q_1k|q_2l|} + \frac{1}{N}\sum_{k=1}^{N}\Big(G^{(g)}_{|q_1kq_1q_2|} + G^{(g)}_{|q_2kq_2q_1|} + G^{(g)}_{|q_1kq_2k|}\Big)$$
$$+ \frac{1}{N}\sum_{k=1}^{N}\Big(G^{(g-1)}_{|q_1k|q_2|q_2|} + G^{(g-1)}_{|q_2k|q_1|q_1|}\Big) + G^{(g-1)}_{|q_1q_2q_2|q_2|} + G^{(g-1)}_{|q_2q_1q_1|q_1|}$$
$$+ \sum_{g_1+g_2=g-1} G^{(g_1)}_{|q_1|q_2|} G^{(g_2)}_{|q_1|q_2|} + G^{(g-2)}_{|q_1|q_1|q_2|q_2|}\,.$$

- **Lesson** for theoretical physicists: are we looking for the wrong objects in complicated QFT? Enormous simplification after reordering!

- Formulae for more boundaries can be written down, but are extremely lengthy. The upper simplification may be the reason for the algebraic beauty and the appearance of exact solvability of the model.

For the pair of pants and torus topologies, the solution strategy produces more than TR allows:

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Proposition [BHW20a]

$$\Omega_3^{(0)}(u,v,z) = \frac{\partial^3}{\partial R(u)\partial R(v)\partial R(z)}$$
$$\Big[\frac{\lambda(\frac{1}{v+u}+\frac{1}{v-u})}{R'(u)R'(-u)(z+u)} + \frac{\lambda(\frac{1}{u+v}+\frac{1}{u-v})}{R'(v)R'(-v)(z+v)}$$
$$+\sum_{i=1}^{2d}\frac{\lambda(\frac{1}{v+\beta_i}+\frac{1}{v-\beta_i})(\frac{1}{u+\beta_i}+\frac{1}{u-\beta_i})}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)}\Big]$$

- Cannot be created via usual **TR**. Does a generic structure of the remainder exist?
- The recursion kernel only contains the standard bifferential B. *Not symmetric!*

symmetric!

- For higher genera, also the **zero** as **fixed point** of the global involution of the spectral curve becomes important:

Proposition [BHW20a]

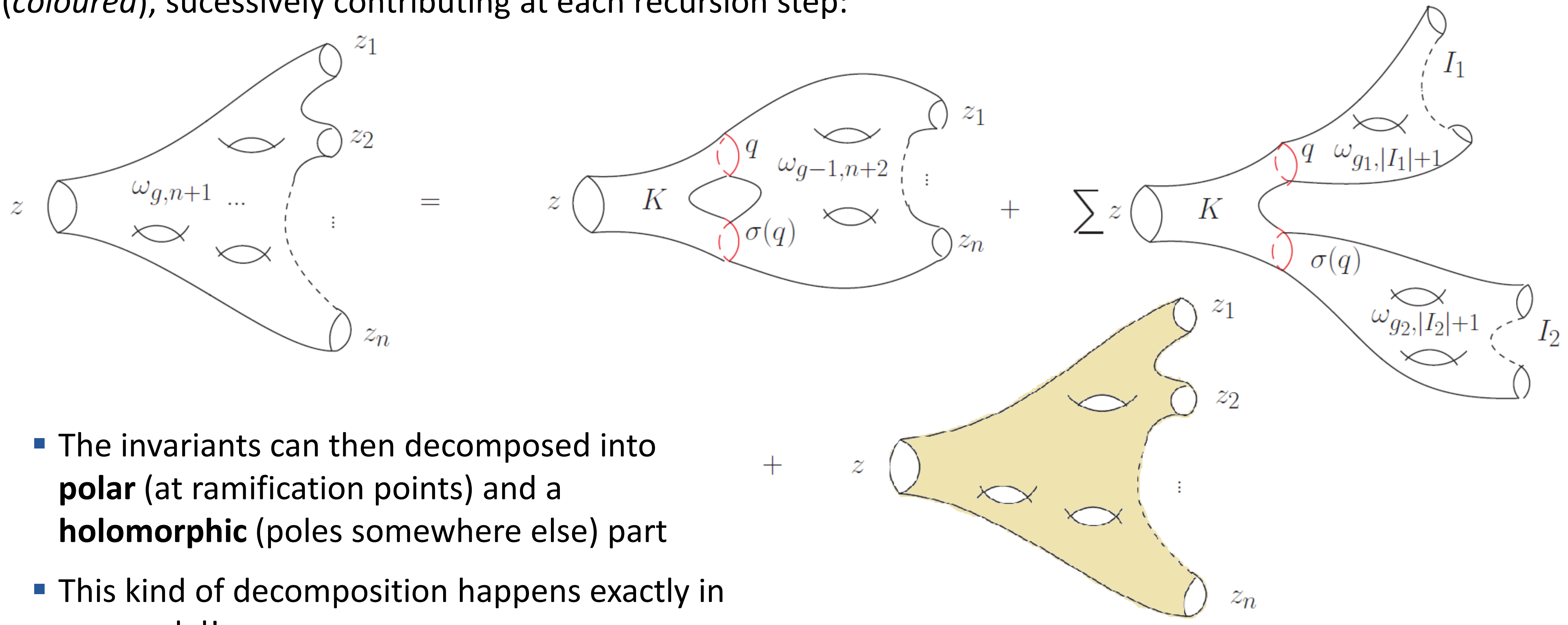
The surplus structure in $\omega_{1,1}(z)$ reads:

$$\frac{\lambda R''(0)}{16R'(0)^3z^2} - \frac{\lambda}{8R'(0)^2z^3}$$

Blobbed topological recursion enlarges the class of models covered by a TR-like structure.

A model does not perfectly fit into the TR picture?

- One could rescue the recursion by allowing for an infinite set of initial data (*coloured*), sucessively contributing at each recursion step:



- The invariants can then decomposed into **polar** (at ramification points) and a **holomorphic** (poles somewhere else) part
- This kind of decomposition happens exactly in our model!

$$\omega_{g,n+1}(\ldots,z) = \mathcal{P}_z\omega_{g,n+1}(\ldots,z) + \mathcal{H}_z\omega_{g,n+1}(\ldots,z)$$

Let us cite the main theorem for the planar blob structure:

Theorem [HW21]

- The polar and holomorphic part of $\omega_{0,|I|+1}(I,z)$ with $I=\{u_1,\dots,u_m\}$ decompose as follows:

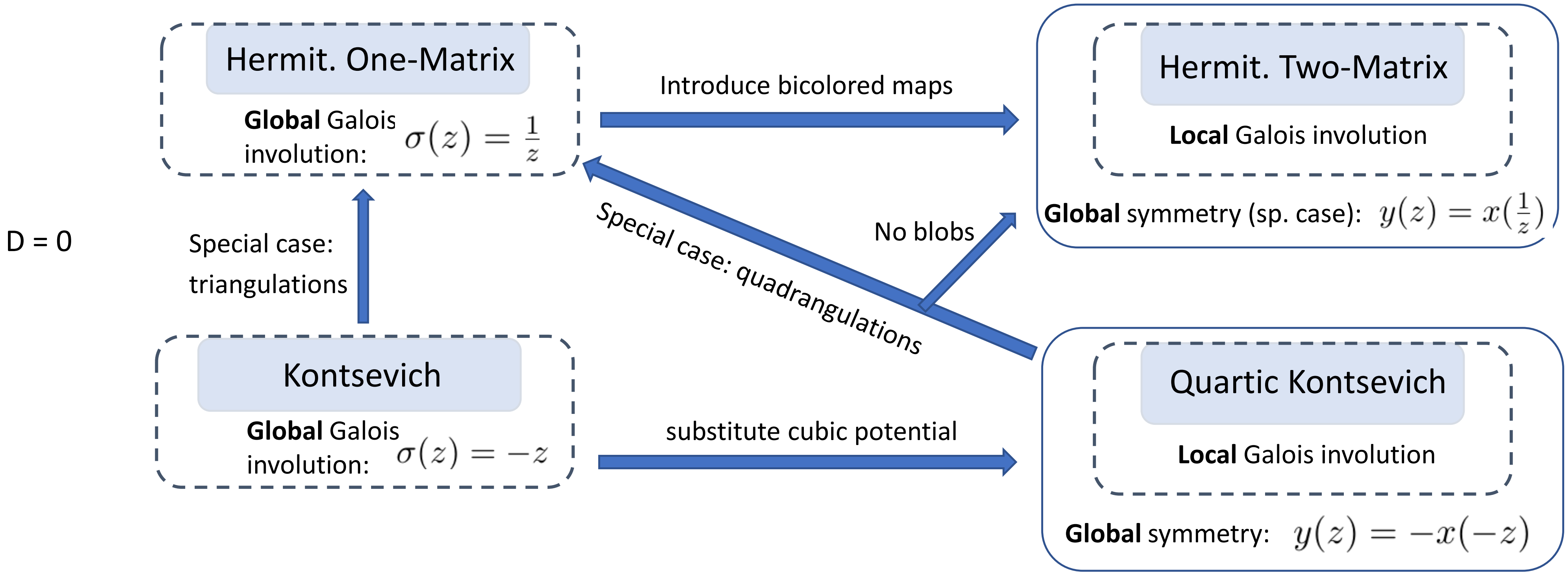
$$\omega_{0,|I|+1}(I,z)=\sum_{i=1}^{r}\operatorname*{Res}_{q\to\beta_i}K_i(z,q)\sum_{I_1\uplus I_2=I}\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q))$$
$$-\sum_{k=1}^{m}d_{u_k}\Big[\operatorname*{Res}_{q\to -u_k}\sum_{I_1\uplus I_2=I}\tilde{K}(z,q,u_k)d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,q)\big)\Big]$$

with an analogous holomorphic kernel

$$K_i(z,q):=\frac{\frac{1}{2}\Big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\Big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))},\qquad \tilde{K}(z,q,u):=\frac{\frac{1}{2}\Big(\frac{dz}{z-q}-\frac{dz}{z+u}\Big)}{dx(q)(y(q)+x(u))}$$

- This explicit formula, remarkably similar to usual TR, is exceptional in examples for BTR
- Linear and quadratic loop equations are fulfilled

Our model in the realm of matrix models: in a nutshell

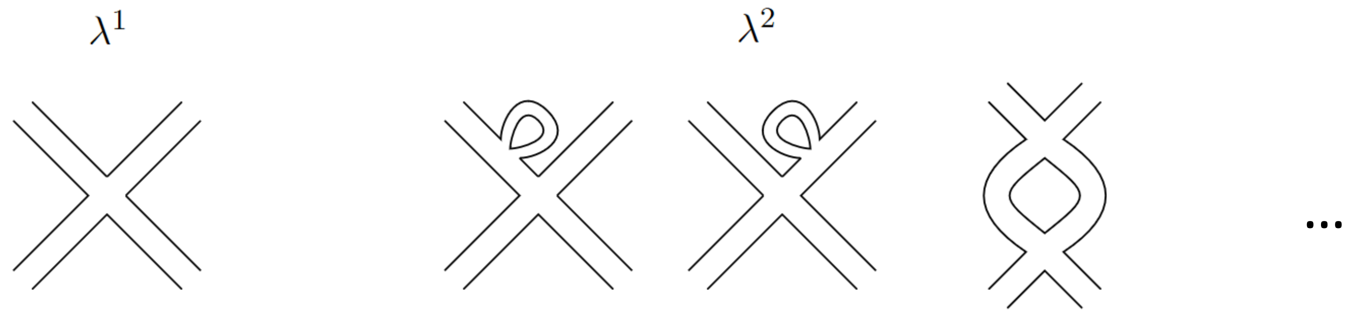


- The special cases occur for a d-fold degenerate eigenvalue, shrinking the external field to a scalar
- One naturally observes a direct correspondence to Hermitian matrix models

Perturbation theory in two interesting limits.

The correlation functions possess a perturbative expansion into ribbon graphs/fat Feynman graphs, e.g.

$G^{(0)}_{|abcd|}$:



$d = 1$ *(scalar external field)*

- In this **combinatorial limit**, the coefficients in the perturbative expansion simply count the number of Feynman graphs at each order
- One enters the realm of purely enumerative geometry
- The correlation functions create fully simple maps (quadrangulations only)

$d \to \infty$ *(continuum limit)*

- In this **QFT limit**, the loops in the diagrams have to be integrated out over all energies
- The integrals may possess **divergencies** that have to be cancelled (field and mass **renormalization**)
- **Zimmermann's forest formula** for ribbon graphs gives a recipe to turn any diagram convergent

Application 1: Limit of "external scalar".

- Consider the starting point of the recursion, one boundary: interprete both sides in terms of maps

$$\Omega_{q_1}^{(g)} = \frac{1}{N}\sum_{k=1}^{N} G_{|q_1 k|}^{(g)} + G_{|q_1|q_1|}^{(g-1)}$$

*Generating function of **ordinary** quadrangulations*

- Maps counted by the Hermitian 1-matrix model
- Equivalence to QKM, since the closed maps/free energies are unique and the loop insertion operators creating the first boundary are equivalent

lhs: e.g. ordinary tori decaying into 2 categories

1) Boundary simple, edges not identified
2) Vertices identified, boundary: non-trivial cycle

*Generating function of **fully simple** quadrangulations*

- Maps counted by the Hermitian 1-matrix model exchanging the role of *x* and *y* in the spectral curve!
- Subset: only boundaries where no more than two belonging edges are incident to a vertex

rhs: exactly the 2 categories:

1) Fully simple tori
2) Fully simple cylinders with boundary length (1,1)

For **g=1** this proposition was known from the studies of fully simple quadrangulations (Borot, Garcia Failde). Our starting condition naturally **generalize it to arbitrary genus**, derive starting condition only from enumerative geometry!

Application 2: The QFT limit and number theory (questions for the future)

- Remember: continuous spectrum of the external field demands to integrate out any internal variable in the Feynman graphs
- for n loops, the generic integration has the following form:

$$\mathrm{Hlog}(a, [k_1, \ldots, k_n]) := \int_0^a \frac{dx_1}{x_1 - k_1} \int_0^{x_1} \frac{dx_2}{x_2 - k_2} \ldots \int_0^{x_{n-1}} \frac{dx_n}{x_n - k_n}$$

- Evaluated at concrete boundaries, multiple zeta values arise from the expansion of the correlation functions:

$$\zeta(a_1, \ldots, a_n) = \sum_{0<k_1<\ldots<k_n} \frac{1}{k_1^{a_1} \ldots k_n^{a_n}}$$

- The invariants of BTR may become generating function for objects of number-theoretical importance!

Serious hindrance: the Dyson-Schwinger equations of the continuous model become nearly unsolvable integral equations. Proving BTR for this limit is a far more complicated challenge

Thank you for your attention! Questions?