Resurgence analysis of the WRT-TQFT

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The WRT Invariant $\tau_k(M, K)$

Reshetikhin and Turaev defined for each $k \in \mathbb{N}$ a TQFT τ_k. For a closed 3-mfd M with a knot K, colored by a representation λ we have the WRT invariant

$$\tau_k(M, K, \lambda) \in \mathbb{C}$$

which is a model for Witten's $\mathrm{SU}(2)$ quantum Chern-Simons theory

$$\mathrm{Z}_k(M, K, \lambda) = \int_{\mathcal{A}_{\mathrm{SU}(2)}/\mathcal{G}_{\mathrm{SU}(2)}} \exp\left(\frac{ki}{4\pi}\,\mathrm{CS}(a)\right) \mathrm{tr}\,\lambda\left(\mathrm{Hol}_K(a)\right) \mathcal{D}a.$$

The WRT-TQFT is a full $2+1$ dimensional TQFT, which can be studied using modular tensor categories, conformal field theory or quantization of moduli spaces of flat connections on surfaces with labeled punctures.

Conjectures about τ_k and resurgence as a Rosetta stone

Main conjectures concerning the WRT-TQFT:

1. The asymptotic expansion conjecture: relating τ_k to classical Chern-Simons theory.

$$\tau_k(M) \sim_{k\to\infty} \sum_{\theta\in \mathrm{CS}_{\mathrm{SU}(2)}} \exp(2\pi i k\theta) k^{d_\theta} b_\theta (1 + Z_\theta(k^{-1})).$$

where $Z_\theta(k^{-1}) = c_\theta^1 k^{-1} + c_\theta^2 k^{-2} + \ldots$.

Main conjectures concerning the WRT-TQFT:

1. The asymptotic expansion conjecture: relating τ_k to classical Chern-Simons theory.
2. The volume conjecture: relating τ_k to hyperbolic geometry. **Kashaev's** original volume conjecture (in the MM formulation):

$$\lim_{k\to\infty} \frac{1}{k} \log\left(\frac{|\tau_k(S^3, K, \lambda = k+2)|}{|\tau_k(S^3, U, \lambda = k+2)|}\right) = \frac{1}{2\pi}\mathrm{Vol}(S^3 - K)$$

Conjectures about τ_k and resurgence as a Rosetta stone

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1. The asymptotic expansion conjecture: relating τ_k to classical Chern-Simons theory.
2. The volume conjecture: relating τ_k to hyperbolic geometry.
3. Integrality and categorification of τ_k: The GPPV invariant of a 3-mfd. M (defined as a string theory BPS index) with a spin^c structure a is an integer power series

$$\hat{Z}_a(M;q) \in \mathbb{Z}[[q]]_0.$$

Conjecture: there exists a homology theory $\mathrm{H}_{\bullet,\bullet}(M;a)$ s.t.

$$\hat{Z}_a(M;q) = \sum_{i,j} (-1)^i q^j \dim\left(\mathrm{H}_{i,j}(M;a)\right).$$

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3. Integrality and categorification of τ_k: The GPPV invariant of a 3-mfd. M (defined as a string theory BPS index) with a $\mathrm{spin}^{\mathrm{c}}$ structure a is an integer power series

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The asymptotic expansion of the TQFT τ_k, the volume conjecture and $\hat{\mathrm{Z}}_a(q)$ are **connected via resurgence.**

Start with:
The asymptotic expansion conjecture: relating τ_k to classical Chern-Simons theory.

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where

$$Z_\theta(k^{-1}) = c_\theta^1 k^{-1} + c_\theta^2 k^{-2} + \dots$$

in general are **divergent series**.

Resummation of divergent power series

- **Divergent series**: In mathematical physics divergent series are common and many examples comes from path integrals.

Resummation of divergent power series

- **Divergent series**: In mathematical physics divergent series are common and many examples comes from path integrals.
- **Problem**: given a divergent power series

$$\varphi(z) \in z^{-1}\mathbb{C}[[z^{-1}]].$$

we want to construct a holomorphic function

$$\widehat{\varphi} \in \mathcal{O}(D)$$

having φ as an asymptotic expansion, i.e. $\forall m \in \mathbb{N}$

$$\widehat{\varphi}(z) = \varphi_0 z^{-1} + \cdots + \varphi_m z^{-m-1} + O(z^{-m-2}).$$

- **Borel-Laplace resummation** is a solution to this problem.

The Borel transform $\mathcal{B}$

Definition 1

The Borel transform

$$\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]$$

is the $\mathbb{C}$-linear extension of

$$\mathcal{B}\left(z^{-\alpha-1}\right) = \frac{\zeta^{\alpha}}{\Gamma(\alpha+1)} = \frac{\zeta^{\alpha}}{\alpha!}.$$

Here Γ is the Gamma function, which for $\mathrm{Re}(x) > 0$ is defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \, \mathrm{d}\, t.$$

The Borel transform $\mathcal{B}$ as the inverse of the Laplace transform $\mathcal{L}$

- **The Laplace transform $\mathcal{L}$:** Let $\gamma \subset \mathbb{C}$ be an oriented countour. Let g be a holomorphic function. Define

$$\mathcal{L}_\gamma(g)(z) = \int_\gamma \exp(-z \cdot \zeta) g(\zeta) \; \mathrm{d}\,\zeta.$$

Proposition 1

For all $m \in \mathbb{N}$ one has

$$\mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(z^{-m-1}) = z^{-m-1},$$
$$\mathcal{B} \circ \mathcal{L}_{\mathbb{R}_+}(\zeta^m) = \zeta^m.$$

Borel-Laplace resummation $\varphi \mapsto \mathcal{L} \circ \mathcal{B}(\varphi)$

Proposition 2

Let $\varphi(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$. Assume $\mathcal{B}(\varphi)(\zeta)$ extends to an analytic function of appropriate bound along $\gamma(\theta) = \exp(i\theta)\mathbb{R}_+$. Consider

$$\widehat{\varphi}(z) \overset{\text{def.}}{=} \mathcal{L}_{\gamma(\theta)} \circ \mathcal{B}(\varphi)(z) = \int_{\gamma(\theta)} e^{-\zeta z} \mathcal{B}(\varphi)(\zeta) \, \mathrm{d}\,\zeta.$$

Borel-Laplace resummation $\varphi \mapsto \mathcal{L} \circ \mathcal{B}(\varphi)$

Proposition 2

Let $\varphi(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$. *Assume* $\mathcal{B}(\varphi)(\zeta)$ *extends to an analytic function of appropriate bound along* $\gamma(\theta) = \exp(i\theta)\mathbb{R}_+$. *Consider*

$$\widehat{\varphi}(z) \overset{\text{def.}}{=} \mathcal{L}_{\gamma(\theta)} \circ \mathcal{B}(\varphi)(z) = \int_{\gamma(\theta)} e^{-\zeta z}\mathcal{B}(\varphi)(\zeta)\ \mathrm{d}\,\zeta.$$

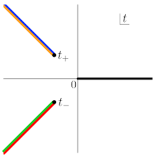
The function $\widehat{\varphi}(z)$ *is analytic on Re*$(z\exp(i\theta)) > 0$ *and has* $\varphi(z)$ *as Poincaré asymptotic expansion*

$$\widehat{\varphi}(z) \sim_{|z|\to\infty} \varphi(z) \in z^{-1}\mathbb{C}[[z^{-1}]].$$

Picard-Lefshetz theory and resurgence

- **Picard-Lefshetz theory and resurgence:** Let $f \in \mathcal{O}(\mathbb{C}^d)$, let $\omega \in \Omega^d_{\mathrm{Hol}}(\mathbb{C}^d)$, let Δ be a PL-thimble and consider

$$\mathrm{I}_\Delta(\lambda) = \int_\Delta \exp(-\lambda f(z))\, \omega(z)$$



Think of the t-plane as the set of values of f with a discrete set of critical values (here $0, t_+, t_-$) with curves γ emanating from them, along which the exponential factor is decaying.

Illustration of a Picard-Lefschetz thimble $\Delta(\sigma, \gamma)$

Below we illustrate a Picard-Lefshetz thimble $\Delta(\sigma, \gamma)$ foliated by vanishing cycles $\sigma(t) \in \mathrm{H}_*(f^{-1}(t), \mathbb{Z})$ which are parallel (w.r.t. the Gauss-Manin connection) along a curve $\gamma \subset \mathrm{Im}(f)$

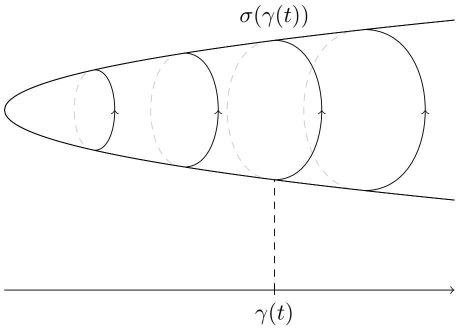


Figure: Thimble $\Delta(\sigma, \gamma)$ in $d = 2$.

Picard-Lefshetz theory and resurgence: Let $f \in \mathcal{O}(\mathbb{C}^d)$, let $\omega \in \Omega^d_{\mathrm{Hol}}(\mathbb{C}^d)$, let Δ be a PL-thimble and consider

$$\mathrm{I}_\Delta(\lambda) = \int_\Delta \exp(-\lambda f(z))\, \omega(z).$$

Let $\tilde{\mathrm{I}}_\Delta \in \xi^{-1}\mathbb{C}[[\xi^{-1}]]$ be such that we have the asymptotic expansion

$$\mathrm{I}_\Delta(\lambda) \sim \exp(-\lambda f(z_\Delta))\lambda^{d_\Delta}(1 + \tilde{\mathrm{I}}_\Delta(\lambda^{-1})).$$

The aim of Écalle's theory of resurgence is to decode the information contained in the divergent series

$$\tilde{\mathrm{I}}_\Delta \in \xi^{-1}\mathbb{C}[[\xi^{-1}]]$$

and determine the properties of the Borel resummation of them as meromorphic multi-value functions and in turn to recover from them the actual integrals $\mathrm{I}_\Delta(\lambda)$.

Picard-Lefshetz theory and resurgence

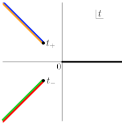
In particular, when we turn the argument of the expansion parameter λ around, then the γ's emanating from some critical point will hit some other critical point and at this point the corresponding I_Δ jumps to

$$\mathrm{I}_\Delta \mapsto \mathrm{I}_\Delta + n(\Delta, \Delta')\,\mathrm{I}_{\Delta'},$$

hence we see that

> $\mathrm{I}_{\Delta'}$ resurges with a multiplicity factor $n(\Delta, \Delta')$ in I_Δ.

Écalle has developed his Alien calculus precisely to determine these jumping or wall crossing phenomenon directly from the divergent series $\tilde{\mathrm{I}}_\Delta$.



Resurgence in TQFT: The analogy between $\mathrm{I}_{\Delta}(\lambda)$ and the partition function of $\mathrm{SU}(2)$ Chern-Simons theory

$$\mathrm{Z}(k) = \int_{\mathcal{A}_{\mathrm{SU}(2)}/\mathcal{G}_{\mathrm{SU}(2)}} \exp\left(\frac{ki}{4\pi}\mathrm{CS}(\mathrm{a})\right)\mathcal{D}a$$

was used by Witten, Garoufalidis and Gukov-Marino-Putrov, suggesting deep connections to $\mathrm{SL}(2,\mathbb{C})$ Chern-Simons theory, by thinking of

$$\mathcal{A}_{\mathrm{SU}(2)}/\mathcal{G}_{\mathrm{SU}(2)} \subset \mathcal{A}_{\mathrm{SL}(2,\mathbb{C})}/\mathcal{G}_{\mathrm{SL}(2,\mathbb{C})}.$$

as a cycle and decomposing it into

"middle dimensional Picard-Lefshetz thimbles"

in this path integral setting.

So back to the asymptotic expansions of WRT-invariants:
The asymptotic expansion conjecture: relating τ_k to classical Chern-Simons theory.

$$\tau_k(M) \sim_{k\to\infty} \sum_{\theta \in \mathrm{CS}_{\mathrm{SU}(2)}} \exp(2\pi i k\theta) k^{d_\theta} b_\theta (1 + Z_\theta(k^{-1})).$$

where

$$Z_\theta(k^{-1}) = c_\theta^1 k^{-1} + c_\theta^2 k^{-2} + \dots$$

in general are **divergent series**.

The Resurgence Conjecture:

1. Borel Resummability: The series Z_θ are Borel resummable.
2. Generalised Volume Conjecture: The set of poles Ω of the meromorphic functions $\mathcal{B}(Z_\theta)$ satisfies that

$$\mathrm{CS}_{\mathbb{C}}(M) = \frac{i}{2\pi}\Omega \mod \mathbb{Z}.$$

3. Wall Crossing: The meromorphic functions $\mathcal{B}(Z_\theta)$ satisfies and are in part determined by a Wall crossing structure.
4. The $\hat{Z}$-Conjecture: The $\hat{\mathrm{Z}}_a$ GPPV invariants can be obtained from the functions $\mathcal{B}(Z_\theta)$ by a Laplace type transform.
5. The Radial Limit Conjecture: The WRT invariant τ_k can be (re)-obtained from $\hat{\mathrm{Z}}_0$ as a limit

$$\tau_k = \frac{1}{\sqrt{k}} \lim_{q \to e^{2\pi i/k}} \hat{\mathrm{Z}}_0(q).$$

The radial limit conjecture of Gukov, Pei, Putrov and Vafa

Conjecture: Let Y be a closed oriented rational homology sphere. Set $T = \mathrm{spin}^c(Y)/\mathbb{Z}_2$. For every $a \in T$, there exists invariants

$$\Delta_a \in \mathbb{Q},\ c \in \mathbb{Z}_+,\ \hat{\mathrm{Z}}_a(q) \in 2^{-c} q^{\Delta_a}\mathbb{Z}[[q]],$$

such that: $\hat{\mathrm{Z}}_a(q)$ is convergent for $|q| < 1$ and for infinitely many k, the radial limits $\lim_{q\to\exp(2\pi i/k)} \hat{\mathrm{Z}}_a(q)$ exists and

$$\tau_k(Y) = \frac{-i}{\sqrt{2k}} \sum_{a,b\in T} e^{2\pi i k\cdot lk(a,a)} S_{a,b} \lim_{q\to\exp(2\pi i/k)} \hat{\mathrm{Z}}_b(q).$$

Here

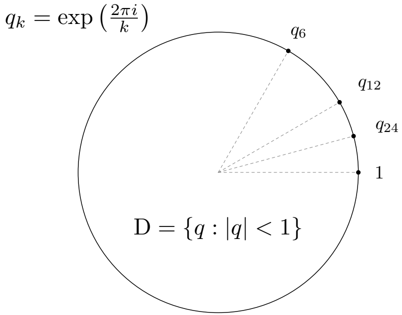
$$S_{a,b} = \frac{e^{2\pi i k\cdot lk(a,b)} + e^{-2\pi i k\cdot lk(a,b)}}{|\mathcal{W}_b||\mathcal{W}_a|\sqrt{|\mathrm{H}_1(Y;Z)|}},$$

and $\mathcal{W}_x$ is the $\mathbb{Z}_2$-stabilizer of x.

Analytic continuation of $\tau(M, q_k)$ to the unit disc $|q| < 1$

Analytic extension: The radial limit conjecture gives a way of extending $\tau(M, q_k)$ to the unit disc. For $H_1(M, \mathbb{Z}) = 0$ it states

$$\tau(M, q_k) = k^{-1/2} \lim_{q \to q_k} \hat{\mathrm{Z}}_0(M; q).$$



WRT invariant: $\tau(M) : \mathbb{N} \to \mathbb{C}$

GPPV invariant: $\hat{\mathrm{Z}}_0(M) : \mathrm{D} \to \mathbb{C}$

Radial limit: $\frac{1}{\sqrt{k}} \hat{\mathrm{Z}}_0(M; q_k) = \tau_k(M)$

Figure: Extension of $\tau_k(M)$.

Plumbed 3-manifolds $\mathbf{Y}_\Gamma$

- **Graph Γ** : Let (Γ, m) be an ordered weighted tree, i.e. Γ is a tree with an ordering of the vertices V and $m : V \to \mathbb{Z}$.

- **Adjacency matrix $\mathbf{M}$:** Let M be the $V \times V$ matrix

$$M_{i,j} = \begin{cases} m_v & \text{if } v_i = v_j = v, \\ 1 & \text{if } v_i \text{and } v_j \text{ are joined by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

  Assume $\det(M) \neq 0$ and M^{-1} is neg. definite.

- **Plumbed manifold $\mathbf{Y}_\Gamma$:** For each $v \in V$ the surgery link L has an unknot component L_v with framing m_v, and L_v is chained together with L_w if v and w are joined by an edge.

Topological invariance and the definition of $\hat{Z}_a(Y; q)$

Definition 2 (Gukov,Pei,Putrov,Vafa)

Let Γ be a plumbing graph. Then there is an explicit definition of

$$\Delta_a \in \mathbb{Q},\ c \in \mathbb{Z}_+,\ \hat{Z}_a(\Gamma; q) \in 2^{-c} q^{\Delta_a} \mathbb{Z}[[q]],$$

for each spin$_c$-structure a on $\mathbf{Y}_\Gamma$.

Theorem 3 (Gukov, Manolescu)

If $\mathbf{Y}_\Gamma = \mathbf{Y}'_\Gamma$ (e.g. Γ and Γ' are related by Neumann moves) then

$$\hat{Z}_a(\Gamma; q) = \hat{Z}_a(\Gamma'; q).$$

Results: (Joint with William Elbaek Mistegaard)

Resurgence Analysis of WRT-Invariants of SF-manifolds

For a Seifert homology sphere $X = \Sigma(p_1, ..., p_n)$ we prove:

1. A decomposition and idenfitication

   $$\pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{R}))\cup\mathcal{M}(\mathrm{SU}(2))) \cong \pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{C}))) \cong \mathrm{CS}_{\mathbb{C}}(X).$$

2. $\hat{\mathrm{Z}}_0(q)$ is a resummation of the Ohtsuki series Z_0

   $$\hat{\mathrm{Z}}_0(q) = \frac{1}{\sqrt{\tau}}\mathcal{L}\circ\mathcal{B}(\mathrm{Z}_0)(1/\tau), \qquad (q = \exp 2\pi i\tau).$$

3. A full asymptotic expansion of $\hat{\mathrm{Z}}_0(q)$ for τ near $1/k$ implying

   $$\tau_k = \frac{1}{\sqrt{k}} \lim_{q\to e^{2\pi i/k}} \hat{\mathrm{Z}}_0(q).$$

4. An identification of the poles Ω of the Borel transform $\mathcal{B}(\mathrm{Z}_0)$

   $$-2\pi i\, \mathrm{CS}^*_{\mathbb{C}}(X) = \Omega + 2\pi i\mathbb{Z}.$$

A Seifert fibered homology 3-sphere $X = \Sigma(p_1, ..., p_n)$

Let $p_1, ..., p_n \in \mathbb{N}, n \geq 3$ be pairwise coprime and consider the Seifert fibered homology 3-sphere with $n \geq 3$ exceptional fibers

$$X = \Sigma(p_1, ..., p_n).$$

Our work builds on work of Lawrence and Rozansky on $\tau_k(X)$ and is inspired by work of Gukov, Marino and Putrov.

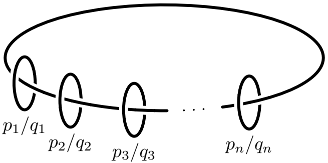


Figure: Surgery link for X.

Complex Chern-Simons theory on X

For $x \in \mathbb{Q}$ let $[x] = x \mod \mathbb{Z}$. Set $P = p_1 \cdots p_n$. We prove

Theorem 4 (A. & Mistegaard)

The Chern-Simons action is injective on $\pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{C})))$ *and*

$$\mathrm{CS}^*_{\mathbb{C}}(X) = \left\{ \left[\frac{-m^2}{4P} \right] : m \in \mathbb{Z} \text{ is divisible by at most } n-3 \text{ of the } p_j\text{'s} \right\}.$$

The natural inclusion $\mathcal{M}(\mathrm{SL}(2,\mathbb{R})) \cup \mathcal{M}(\mathrm{SU}(2)) \to \mathcal{M}(\mathrm{SL}(2,\mathbb{C}))$ *induces an isomorphism on the level of* π_0

$$\pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{R})) \sqcup_{\mathcal{M}(U(1))} \mathcal{M}(\mathrm{SU}(2))) \cong \pi_0(\mathcal{M}(\mathrm{SL}(2,\mathbb{C}))).$$

Introduce the rational function

$$G(z) = \frac{\prod_{j=1}^{n}\left(z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}}\right)}{\left(z^{P} - z^{-P}\right)^{n-2}} = (-1)^n \sum_{m=1}^{\infty} \chi_m z^m \in \mathbb{Z}[[z]].$$

The Borel transform and complex Chern-Simons theory

Introduce the rational function

$$G(z) = \frac{\prod_{j=1}^{n}\left(z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}}\right)}{\left(z^P - z^{-P}\right)^{n-2}} = (-1)^n \sum_{m=1}^{\infty} \chi_m z^m \in \mathbb{Z}[[z]].$$

Theorem 5 (A. & Mistegaard)

1. *Set* $c = \sqrt{2\pi i P}$. *The Borel transform* $\mathcal{B}(\mathrm{Z}_0)$ *is the function*

$$\mathcal{B}(\mathrm{Z}_0)(\zeta) = \frac{4c}{\pi i\sqrt{\zeta}} G\left(\exp\left(\frac{c\sqrt{\zeta}}{P}\right)\right).$$

2. *Let* Ω *be the set of poles of* $\mathcal{B}(\mathrm{Z}_0)$. *Then*

$$\mathrm{CS}^*_{\mathbb{C}}(X) = \frac{i}{2\pi}\Omega \mod \mathbb{Z}.$$

The resurgence formula $\hat{\mathrm{Z}}_0 = \mathcal{L} \circ \mathcal{B}(\mathrm{Z}_0)$

Theorem 6 (A. & Mistegaard)

Set $q = \exp(2\pi i\tau), \tau \in \mathbb{H}$. *We have*

$$\hat{\mathrm{Z}}_0(q) = \frac{1}{\sqrt{\tau}} \int_{\Gamma} \exp\left(-\frac{\xi}{\tau}\right) \mathcal{B}(\mathrm{Z}_0)(\xi) \, \mathrm{d}\,\xi = \sum_{m=1}^{\infty} \chi_m q^{\frac{m^2}{4P}}.$$

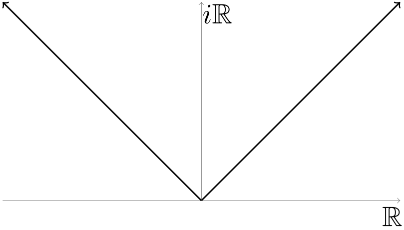


Figure: The integration contour Γ.

The asymptotic expansion of $\hat{\mathrm{Z}}_0$

For small $t > 0$ set $q_{k,t} = \exp\left(\frac{2\pi i}{k - i\frac{2Pt}{\pi}}\right) \in \mathfrak{h}$.

Theorem 7 (A. & Mistegaard)

For each $\theta \in \mathrm{CS}^*_{\mathbb{C}}(X)$ $\exists$ a polynomial in k of degree at most $n-3$ with coefficients in formal power series without constant terms

$$\check{\mathrm{Z}}_\theta(k,t) \in t \cdot \mathbb{Q}[\pi i, k][[t]]$$

giving an asymptotic expansion for small t and fixed even k

$$\hat{\mathrm{Z}}_0(X; q_{k,t}) \underset{t \to 0}{\sim} \tau_k(X) + \sum_{\theta \in \mathrm{CS}^*_{\mathbb{C}}(X)} e^{2\pi i k \theta} \check{\mathrm{Z}}_\theta(k,t).$$

In particular the radial limit conjecture holds for X.

A resurgence formula for τ_k

Our proof of the asymptotic expansion is based on the following resurgence lemma where Ω is the set of poles of $\mathcal{B}(\mathrm{Z}_0)$

Lemma 8 (A. & Mistegaard)

$$\hat{\mathrm{Z}}_0(q) = \mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(\mathrm{Z}_0)\left(\frac{1}{\tau}\right) + \sum_{\omega\in\Omega} \mathit{Res}_{y=\omega}(e^{-y/\tau}\mathcal{B}(\mathrm{Z}_0)(y)).$$

As a corollary of this and the radial limit theorem, we obtain the following resurgence formula for the WRT quantum invariant

Corollary 9 (A. & Mistegaard)

$$\tau_k = \mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(\mathrm{Z}_0)(k) + \sum_{\omega\in\Omega} \mathit{Res}_{y=\omega}(e^{-ky}\mathcal{B}(\mathrm{Z}_0)(y)).$$

Hyperbolic surgeries on the figure 8 knot

We now present in more detail the analysis leading to the above results for the hyperbolic 3-manifolds

$$M(\mathbf{4}_1(p/s)) = M_{p/s}$$

with surgery link giving by the figure eight knot with framing p/s.

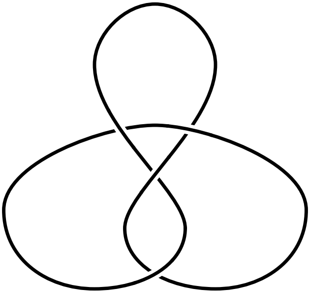


Figure: Figure eight knot $\mathbf{4}_1$

Faddeev's quantum dilogarithm

Recall Faddeev's quantum dilogarithm with parameter $\gamma = \pi/k \in (0,1)$

$$S_\gamma(z) = \exp\left(\frac{1}{4}\int_{\widetilde{C}} \frac{e^{zy}}{\sinh(\pi y)\sinh(\gamma y)y}\ \mathrm{d}\,y\right).$$

for $|\mathrm{Re}(z)| < \gamma + \pi$, and

$$\widetilde{C} = (-\infty, -1/2) \cup \Delta \cup (1/2, \infty)$$

where Δ is a sufficiently small half-circle from $-\epsilon$ to ϵ in the upper half plane ($\epsilon > 0$ small enough).

Contour integral formula for the WRT-Invariant

Choose $c, d \in \mathbb{Z}$ with $rd - cs = 1$. Define

$$\chi_{n,k}(x,y) = \sin\left(\frac{\pi}{s}(x - nd)\right) e^{2\pi i k\left(\frac{dn^2}{s} + \frac{r}{4s}x^2 - \frac{n}{s}x - xy\right)}$$
$$\times \frac{S_\gamma\left(-\pi + 2\pi(x-y)\right)}{S_\gamma\left(-\pi + 2\pi(x+y)\right)} \cot(\pi k x)\tan(\pi k y).$$

Joint with Hansen we proved

$$\tau_k(M_{r/s}) = \nu k q^\mu \sum_{n \in \mathbb{Z}/|s|\mathbb{Z}} \int_{C_1(k)\times C_2(k)} \chi_{n,k}(x,y)\ \mathrm{d}\,y\,\mathrm{d}\,x$$

where $\nu, \mu \in \mathbb{C}^*$ and $C_1(k)$ is a simple closed contour which encircles the set $\{m/k : m = 1, 2, ..., k-1\}$, and $C_2(k)$ is a simple closed contour encircling $\{(m + 1/2)/k : m = 0, 1, ..., k-1\}$.

Contour integral formula for the WRT-Invariant

Define

$$\tilde{\chi}_{n,k}(x,y) = \sin\left(\frac{\pi}{s}(x-nd)\right) e^{2\pi i k\left(\frac{dn^2}{s}+\frac{r}{4s}x^2-\frac{n}{s}x-xy\right)} \times \frac{S_\gamma\left(-\pi+2\pi(x-y)\right)}{S_\gamma\left(-\pi+2\pi(x+y)\right)} s(x,y).$$

where

$$s(x,y) = \mathrm{sign}(\mathrm{Im}(x))\mathrm{sign}(\mathrm{Im}(\bar{y}))i^2$$

Theorem 10 (A. & Mistegaard)

$$\left|\int_{C_1(k)\times C_2(k)} \left(\chi_{n,k}(x,y)-\tilde{\chi}_{n,k}(x,y)\right)\,\mathrm{d}\,y\,\mathrm{d}\,x\right| = O\left(\frac{1}{k}\right)$$

This follows since $|\cot(\pi k x)\tan(\pi k y) - s(x,y)|$ decays exponentially in $|\mathrm{Im}(x)|, |\mathrm{Im}(y)| > \frac{1}{\pi k}$. But it is technical and involves understanding good estimates for S_γ and Li_2.

Semi-classical expansion of the quantum dilog

The semiclassical asymptotics of S_γ is given by Euler's dilogarithm:
For $\zeta \in \{\mathrm{Re}(z) < \pi\}$ we have

$$S_\gamma(\zeta) = \exp\left(\frac{k}{2\pi i}\,\mathrm{Li}_2\left(-e^{i\zeta}\right) + I_\gamma(\zeta)\right),$$

$$I_\gamma(\zeta) = \frac{1}{4}\int_{\tilde{C}} \frac{e^{z\zeta}}{z\sinh(\pi z)}\left(\frac{1}{\sinh(\gamma z)} - \frac{1}{\gamma z}\right)\,\mathrm{d}\,z.$$

This leads us to the following phase functions indexed by
$\alpha, \beta \in \{0,1\}$ and $n \in \mathbb{Z}/|s|\mathbb{Z}$

$$\Phi_n^{\alpha,\beta}(x,y) = \frac{\mathrm{Li}_2(e^{2\pi i(x+y)}) - \mathrm{Li}_2(e^{2\pi i(x-y)})}{4\pi^2} - \frac{dn^2}{s} + (-\frac{r}{4s}x + \frac{n}{s} + y + \alpha + \beta)x + y(\alpha - \beta).$$

Identification of classical complex Chern-Simons values

Theorem 11 (A. & Hansen)

There exists a surjection

$$(x, y) \mapsto [\rho_{x,y}]$$

from the set of critical points (x, y) *of the phase functions* $\Phi_n^{\alpha,\beta}$ *with* $x \notin \mathbb{Z}$ *onto* $\mathcal{M}^*(M_{r/s}, \mathrm{SL}(2, \mathbb{C}))$. *Moreover, we have that*

$$\Phi_n^{\alpha,\beta}(x, y) = \mathrm{S}_{\mathrm{CS}}([\rho_{x,y}]) \mod \mathbb{Z}.$$

A resurgence Theorem

Based on generalizations of resurgence results for Laplace integrals due to Malgrange and Pham we prove

Theorem 12 (A. & Mistegaard)

There exists resurgent power series

$$\{\mathrm{Z}_\theta(x)\}_{\theta\in\mathrm{CS}} \subset x^{-1}\mathbb{C}[[x^{-1}]]$$

giving a full asymptotic expansion

$$\tau_k(M_{1/p}) \sim_{k\to\infty} k \sum_{\theta\in\mathrm{CS}} e^{2\pi i k\theta}\,\mathrm{Z}_\theta(k).$$

Each Borel transform $\mathcal{B}(\mathrm{Z}_\theta)$ is resurgent with singularities

$$\Omega(\theta) = -2\pi i\,\mathrm{CS}_{\mathbb{C}} + 2\pi i\theta + 2\pi i\mathbb{Z}.$$

Resurgence Analysis of Meromorphic Transformations

Nov. 2. arXiv:2011.01110

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- $W \subset \mathbb{C}$ open connected subset
- $K \in \mathcal{M}(W \times \mathbb{C})$ meromorphic kernel.

$$K_w = K|_{w \times \mathbb{C}} \quad P_{K_w} = \text{ pol divisor of } K_w$$

- $\Gamma_w \subset \mathbb{C} - P_{K_w}$ contour
- $\gamma \in U \subset \mathbb{C}$, open connected subset $\quad 0 \in \bar{U} - U$
- $f \in \mathcal{M}(\mathbb{C}) \quad f_\gamma(z) = f(\gamma z) \quad \Gamma_w \subset \mathbb{C} - (P_{K_w} \cup P_{f_\gamma})$

$$g_\gamma(w) = \int_{\Gamma_w} K(w,z) f_\gamma(z) dz$$

$$A^\gamma_{K,\Gamma} : \mathcal{M}_{K,\Gamma}(\mathbb{C}) \to \mathcal{O}(W), \quad A^\gamma_{K,\Gamma}(f) = g_\gamma$$

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Want to understand the asymptotics of the transforms $A^\gamma_{K,\Gamma}$ as $\gamma \to 0$.

Ex: Fourier transform, Laplace transform, Mellin transform, ...

Faddeev's quantum dilogarithm S_γ

We let

$$K_F(w,z) = \frac{e^{wz}}{\sinh(\pi z)z} \text{ and } f_F(z) = \frac{1}{\sinh(z)}$$

and

$$S_\gamma(w) = \exp(\frac{1}{4}g_\gamma^F(w)) \quad \forall (w,\gamma) \in \tilde{W}^F \times \tilde{U}^F$$

where

$$g_\gamma^F(w) = \int_{\mathbb{R}+i\epsilon} \frac{e^{wz}}{\sinh(\pi z)z} \frac{1}{\sinh(\gamma z)} dz$$

and

$$\tilde{W}^F = \{w \in \mathbb{C} \mid |\mathrm{Re}(w)| < \pi + |\mathrm{Re}(\gamma)|\}, \quad \tilde{U}^F = \{\gamma \in \mathbb{C} \mid \mathrm{Re}(\gamma) > 0\}$$

where $S_\gamma \in \mathcal{M}(\mathbb{C})$ is Faddeev's quantum dilogarithm.

The Gamma function Γ

We now let

$$K_\Gamma(w,z) = \frac{ie^{(w-1)\log(-z)}}{2\sin(\pi z)} \text{ and } f_\Gamma(z) = e^{-z}$$

and

$$\Gamma_w^\Gamma = \{z \in \mathbb{C} \mid d(z, \mathbb{R}_+) = \epsilon\}$$

oriented from $\infty + i\epsilon$ to $\infty - i\epsilon$. We then have that

$$\Gamma_\gamma(w) = g_\gamma^\Gamma(w)$$

where Γ_1 is Euler's Gamma function.

One over the Gamma function $\frac{1}{\Gamma}$

We now let

$$K_{\frac{1}{\Gamma}}(w,z) = \frac{1}{2\pi i} e^{-w\log(z)} \text{ and } f_{\frac{1}{\Gamma}}(z) = e^{z}$$

and

$$\Gamma_w^{\frac{1}{\Gamma}} = \{z \in \mathbb{C} \mid d(z,\mathbb{R}_-) = \epsilon\}$$

oriented from $-\infty - i\epsilon$ to $-\infty + i\epsilon$. We then have that

$$\frac{1}{\Gamma_\gamma(w)} = g_\gamma^{\frac{1}{\Gamma}}(w).$$

Riemann zeta function ζ

We set

$$K_\zeta(w,z) = -\frac{\Gamma(1-w)e^{(w-1)\log(z)}}{2\pi i} \text{ and } f_\zeta(z) = \frac{1}{e^z - 1}$$

and

$$\Gamma_w^\zeta = \{z \in \mathbb{C} \mid d(z, \mathbb{R}_+) = \epsilon\}$$

oriented from $-\infty - i\epsilon$ to $-\infty + i\epsilon$. We then have that

$$\zeta_\gamma(w) = g_\gamma^\zeta(w)$$

where $\zeta = \zeta_1$.

Gauss hypergeometric function ${}_2F_1$

We set

$$K_{{}_2F_1}(w,z) = e^{z\log(-w)} \text{ and } f_{{}_2F_1}(z) = \frac{\Gamma(-z)\Gamma(z+a)\Gamma(z+b)}{\Gamma(z+c)}$$

and

$$\Gamma_w^{{}_2F_1} = i\mathbb{R}$$

with the orientation induced from the usual one on $\mathbb{R}$. Then

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1^\gamma(a,b;c;w) = \frac{1}{2\pi i} g_\gamma^{{}_2F_1}(w)$$

with ${}_2F_1^1(a,b;c;w) = {}_2F_1(a,b;c;w)$ Gauss hypergeometric function.

We consider the asymptotic expansion transform.

Assume

$$h_m(w) = \int_{\Gamma_w} K(w,z) z^m dz.$$

exist for all $m \in \mathbb{Z}$.

Then define

$$\tilde{A}^{\gamma,n}_{K,\Gamma} : \mathcal{M}_{K,\Gamma}(\mathbb{C}) \to \mathcal{O}(W)[\gamma^{-1}, \gamma]]$$

given by

$$\tilde{A}^{\gamma,n}_{K,\Gamma}(f)(w) = \sum_{m=-n_0}^{\infty} a_m h_m(w) \gamma^m,$$

where the Laurent series of f at zero is (n_0 pol order of f at zero)

$$f(z) = \sum_{m=-n_0}^{\infty} a_m z^m$$

Assumption on Γ and estimates on K and f

- $\tilde{\Gamma}_w$ deformation of Γ_w in $\mathbb{C} - (P_{K_w} \cup P_{f_\gamma} - \{0\})$
- $\tilde{\Gamma}_w$ consist of d smooth arc segments $\tilde{\Gamma}_{w,j}$, $j = 1, \ldots, d$.
- $\Theta(z) =$ angle between the line through zero and z and tangent line to $\tilde{\Gamma}_{w,j}$ at z.

- $\exists\, b > 0$:

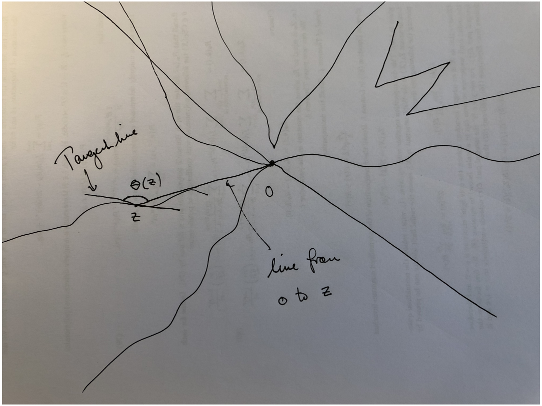
$$\boxed{|\cos(\Theta(z))| \geq b, \quad \forall z \in \tilde{\Gamma}_{w,j},\ \ j = 1, \ldots d.}$$

- $\exists\, \delta > 0$, $c > 0$, $c_w > 0$, $w \in W$ and $\tilde{c}_\gamma > c$, $\gamma \in U$:

$$| K(w,z) | \leq c_w e^{-\delta|z|} |z|^{-k_0} \quad \forall (w,z) \in W \times (\tilde{\Gamma}_w - \{0\})$$

$$|f_\gamma(z)| \leq \tilde{c}_\gamma |\gamma z|^{-n_0} \quad \forall (\gamma, z) \in U \times (\tilde{\Gamma}_w - \{0\}).$$

An example of an allowed $\tilde{\Gamma}_w$ with indication of $\Theta(z)$ at some $z \in \tilde{\Gamma}_w$.



Theorem 13 (A.)

There is an asymptotic expansion in the Poincare sense to all orders

$$A^{\gamma}_{K,\Gamma}(f) \sim \tilde{A}^{\gamma}_{K,\Gamma}(f).$$

Estimate: For all $n \geq k_0 - 1$ *there exist* C_n

$$\mid A^{\gamma}_{K,\Gamma}(f)(w) - \tilde{A}^{\gamma,n}_{K,\Gamma}(f)(w) \mid \leq C_n c_w \tilde{c}_{\gamma} |\gamma|^{n+1} \delta^{-(n-k_0+2)}(n-k_0+2)!$$

for all $(w,\gamma) \in W \times U$.

$$C_{k_0-1} = \frac{d}{bc} c'_{k_0-1}, \quad C_n = \frac{d}{bc\sqrt{n-k_0+1}} c'_n, \quad n \geq k_0$$

$$c'_n = \inf_{0<r<R_f} \frac{1}{r^{n+1}} \left(\frac{c r^{-n_0}}{n-k_0+2} + k(r) \right), \quad n \geq k_0 - 1$$

$$k(r) = \sum_{m=-n_0}^{\infty} |a_m| r^m, \quad r \in (0, R_f),$$

$R_f = \mathbf{dist}(0, P_f - \{0\})$.

Borel resummation of the (divergent) series $\tilde{A}^{\gamma}_{K,\Gamma}(f)$ in γ.

- $g^{-}_{\gamma} \in \mathcal{M}(U)[\gamma]$

$$g^{-}_{\gamma}(w) = \sum_{m=-n_0}^{k_0-1} a_m h_m(w)\gamma^m,$$

- $g^{+} \in \gamma\mathcal{M}(U)[[\gamma]]$

$$g^{+}_{\gamma}(w) = \sum_{m=k_0}^{\infty} a_m h_m(w)\gamma^m.$$

- Borel transform

$$\mathcal{B} : \gamma\mathcal{O}(U)[[\gamma]] \to \mathcal{O}(U)[[\xi]]$$

determined by

$$\mathcal{B}(\gamma^m) = \frac{\xi^{m-1}}{(m-1)!}$$

and formally extended linearly over $\mathcal{O}(U)$.

Let $\mathbb{R}_\theta^+ = e^{i\theta}\mathbb{R}_+$ for $\theta \in \mathbb{R}$

$$\mathcal{L}_\theta(\psi)(\gamma) = \int_{\mathbb{R}_\theta^+} e^{-\xi/\gamma}\psi(\xi)d\xi$$

well defined if

$$|\psi(\xi)| \leq Ce^{\alpha|\xi|}, \forall \xi \in \mathbb{R}_\theta^+$$

and

$$\alpha < \mathrm{Re}(\gamma)\cos\theta + \mathrm{Im}(\gamma)\sin\theta.$$

- $\mathcal{L}_\theta \circ \mathcal{B}(\gamma^m) = \gamma^m \quad \forall m \in \mathbb{Z}_+$ when ever $\mathcal{L}_\theta(\gamma^m)$ is defined.
- Let $\varphi \in \mathcal{M}(\mathbb{C})$ be

$$\varphi(z) = f(z) - \sum_{m=-n_0}^{k_0-1} a_m z^m \quad \varphi_z(\gamma) = \varphi(\gamma z).$$

Theorem 14 (A.)

- *If* $\exists\, \theta, \alpha \in \mathbb{R}$, $0 < \beta < \delta$, $C > 0$ *: Well def.* $\mathcal{L}_\theta^{-1}(\varphi_z)$, $z \in \tilde{\Gamma}_w$,

$$|\mathcal{L}_\theta^{-1}(\varphi_z)(\xi)| \leq Ce^{\alpha|\xi|+\beta|z|}|z|^{k_0} \quad \forall (z,\xi) \in \tilde{\Gamma}_w \times V_{w,\theta}, \quad \forall w \in W$$

$V_{w,\theta} \subset \mathbb{C}$ *open containing the half line* $\mathbb{R}_\theta^+$.

- *Then* g_γ^+ *is Borel summable and*

$$B_w(\xi) = \mathcal{B}(g_\gamma^+(w))(\xi), B_w \in \mathcal{O}(V_{w,\theta})$$

$$B_w(\xi) = \int_{\tilde{\Gamma}_w} K(w,z)\mathcal{L}_\theta^{-1}(\varphi_z)(\xi)dz.$$

$$g_\gamma(w) = g_\gamma^-(w) + \mathcal{L}_\theta(B_w)(\gamma) \quad \forall \gamma \in U_{\theta,\alpha}.$$

Let now the principal part of φ at $p \in P_\varphi$ be denoted

$$\tilde{\varphi}_{z,p}(\gamma) = \sum_{m=1}^{n_\varphi} \frac{b_{p,m}}{(\gamma z - p)^m},$$

where we assume that the pole order is universally bounded by some integer n_φ independent of $z \in \tilde{\Gamma}_w$ and $w \in W$. We observe that if we let

$$\tilde{\varphi}'_{z,p}(\gamma) = \sum_{m=1}^{n_\varphi} \frac{b_{p,m}}{p^m}(-1)^m \sum_{l=0}^{m-1} \binom{m}{l} \left(\frac{z}{p}\right)^{m-l} \frac{\left(\frac{z}{p}\right)^{m-l}}{(\frac{1}{\gamma} - \frac{z}{p})^{m-l}}$$

then

$$\tilde{\varphi}_{z,p}(\gamma) = \sum_{m=1}^{n_\varphi} \frac{b_{p,m}}{p^m}(-1)^m + \tilde{\varphi}'_{z,p}(\gamma).$$

We will assume that there exist an ordering on P_φ such that the series

$$\tilde{\varphi}'_z(\gamma) = \sum_{p \in P_\varphi} \tilde{\varphi}'_{z,p}(\gamma)$$

with respect to that ordering is uniformly absolutely convergent on

$$O_\epsilon = \mathbb{C} - \cup_{p \in P_\varphi} D(p, \epsilon)$$

for some $\epsilon > 0$ and there exist an entire function ψ_z parametriced by $z \in \tilde{\Gamma}_w$ and $w \in W$ such that

$$\varphi_z(\gamma) = \tilde{\varphi}'_z(\gamma) + \psi_z(\gamma) \quad \forall \gamma \in O_\epsilon.$$

We will see in examples that one actually sometimes get that $\psi_z = 0$, but in general we will simply assume that $\mathcal{L}_\theta^{-1}(\psi_z)$ exist for all $z \in \tilde{\Gamma}_w$ and $w \in W$.

Theorem 15 (A.)

Assume that exists $\theta, \alpha \in \mathbb{R}$, $0 < \beta < \delta$ and a constant C_ψ, such that for all $w \in W$ we have that

$$|\mathcal{L}_\theta^{-1}(\psi_z)(\xi)| \le C_\psi e^{\alpha|\xi| + \beta|z|} |z|^{k_0} \quad \forall (z, \xi) \in \tilde{\Gamma}_w \times V_{w,\theta},$$

where $V_{w,\theta} \subset \mathbb{C}$ is an open subset containing the half line $\mathbb{R}_\theta^+$. Assuming further that

$$\int_{\mathbb{R}_\theta^+} \sum_{p \in P_\phi} \sum_{m=1}^{n_\varphi} \frac{|b_{p,m}|}{|p|^{2m-1}} \int_{\tilde{\Gamma}_w} \left| e^{-\xi/\gamma} K(w,z) e^{\frac{z\xi}{p}} \right| \left(\left|\frac{z}{p}\right| + |\xi| \right)^{m-1} |dz||d\xi| < \infty$$

and that

$$\frac{1}{|\gamma|^2} \begin{pmatrix} \mathit{Re}(\gamma) \\ \mathit{Im}(\gamma) \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > \frac{1}{|p|^2} \begin{pmatrix} \mathit{Re}(p) & \mathit{Im}(p) \\ \mathit{Im}(p) & -\mathit{Re}(p) \end{pmatrix} \begin{pmatrix} \mathit{Im}(z) \\ \mathit{Re}(z) \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > 0$$

for all $p \in P_\phi$ and $z \in \tilde{\Gamma}_w$, $w \in W$.

Then we have for all $(w, \xi) \in W \times V_{w,\theta}$ that

$$\begin{aligned} B_w(\xi) \quad = \quad & \sum_{p \in P_\varphi} \left(\int_{\tilde{\Gamma}_w} K(w,z) e^{\frac{z\xi}{p}} dz \right) \sum_{m=1}^{n_\varphi} \frac{b_{p,m}}{p^m} (-1)^m \sum_{l=0}^{m-1} \binom{m}{l} \left(\frac{z}{p}\right)^{m-l} \frac{\xi^{m-l-1}}{(m-l-1)!} \\ & + \int_{\tilde{\Gamma}_w} K(w,z) \mathcal{L}_\theta^{-1}(\psi_z)(\xi) dz. \end{aligned}$$

Resurgence properties and Stokes phenomenon of B_w

- $\mathcal{L}_\theta(B_w)$ constant in θ on sectors where B_w has no poles
- Jump when $\mathbb{R}_\theta^+$ hits poles of B_w
- Jumps $\theta_{J_w} = \{\theta_j \mid j \in J_w\}$ index by a set J_w, $w \in W$.
- Let the jump at θ_j be denote $\Delta_{\theta_j}(\mathcal{L}_\theta(B_w))$.

Theorem 16 (A.)

Under the above assumptions we have that

$$\Delta_{\theta_j}(\mathcal{L}_\theta(B_w)) = 2\pi i \sum_{p \in P_{B_w} \cap e^{i\theta_j}\mathbb{R}_+} \mathrm{Res}_{\xi=p}(e^{-\xi/\gamma} B_w(\xi))$$

provided B_w decays sufficiently fast in a small sector around $\mathbb{R}_\theta^+$, $j \in J_w$, $w \in W$.

We let

$$K_F(w,z) = \frac{e^{wz}}{\sinh(\pi z) z} \text{ and } f_F(z) = \frac{1}{\sinh(z)}$$

and

$$S_\gamma(w) = \exp(\frac{1}{4} g_\gamma^F(w)) \quad \forall (w,\gamma) \in \tilde{W}^F \times \tilde{U}^F$$

where

$$g_\gamma^F(w) = \int_{\mathbb{R}+i\epsilon} \frac{e^{wz}}{\sinh(\pi z) z} \frac{1}{\sinh(\gamma z)} dz$$

and

$$\tilde{W}^F = \{w \in \mathbb{C} \mid |\mathrm{Re}(w)| < \pi + |\mathrm{Re}(\gamma)|\}, \quad \tilde{U}^F = \{\gamma \in \mathbb{C} \mid \mathrm{Re}(\gamma) > 0\}$$

where $S_\gamma \in \mathcal{M}(\mathbb{C})$ is Faddeev's quantum dilogarithm.

We observe that $k_0 = 2$ and $n_0 = 1$ and further that $R_{f_F} = \pi$. Laurent series for f_F convergent in $D(0,\pi)$ is

$$f_F(z) = \sum_{m=0}^{\infty} \frac{2(1-2^{2m-1})B_{2m}}{(2m)!} z^{2m-1} \quad \forall z \in D(0,\pi),$$

where B_{2m} is the $2m$'th Bernoulli number. Let

$$P_\gamma^n(w) = \frac{1}{4}\sum_{m=0}^{n} \frac{2(1-2^{2m-1})B_{2m}}{(2m)!}\gamma^{2m-1}\int_{\mathbb{R}+i\epsilon} \frac{e^{wz}}{\sinh(\pi z)z} z^{2m-1} dz.$$

$$\frac{1}{2i}\mathrm{Li}_2(-e^{iw}) = \frac{1}{4}\int_{\mathbb{R}+i\epsilon} \frac{e^{wz}}{\sinh(\pi z)z^2} dz$$

$$\frac{1}{2i}\left(\frac{\partial}{\partial w}\right)^{2m} \mathrm{Li}_2(-e^{iw}) = \int_{\mathbb{R}+i\epsilon} \frac{e^{wz}}{\sinh(\pi z)z} z^{2m-1} dz$$

$$P_\gamma^n(w) = \frac{1}{2i}\sum_{m=0}^{2n} \frac{2(1-2^{2m-1})B_{2m}}{(2m)!}\left(\frac{\partial}{\partial w}\right)^{2m} \mathrm{Li}_2(-e^{iw}).$$

For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we let

$$\tilde{\Gamma}^F_\theta = e^{i\theta}(-\infty, 0] \cup e^{i\theta}[0, \infty).$$

(so $d^F = 2$)

$$W^F_\theta = \left\{ w \in \mathbb{C} \middle| -\pi\cos\theta + \delta < \begin{pmatrix} \mathrm{Re}(w) \\ \mathrm{Im}(w) \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} < \pi\cos\theta - \delta \right\}$$

and

$$U^F_\theta = \left\{ \gamma \in \mathbb{C} \middle| \theta - \frac{\pi}{2} < \mathrm{Arg}(\gamma) < \frac{\pi}{2} + \theta \right\} = \left\{ \gamma \in \mathbb{C} \middle| \begin{pmatrix} \mathrm{Re}(\gamma) \\ \mathrm{Im}(\gamma) \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > 0 \right\}$$

This definition of U^F_θ guarantees that the poles of f^F_γ, which are all on the imaginary axis, when γ is positive real, never crosses $\tilde{\Gamma}^F_\theta$ as the absolute value of the argument of γ grows from zero to π, not including π. We observe that

$$\mathrm{Re}(\gamma)\cos\theta(\gamma) \neq \mathrm{Im}(\gamma)\sin\theta(\gamma)$$

for all $\gamma \in U^F_\theta$.

Application to Faddeev's quantum Dilog.

Theorem 17 (A.)

We have the following estimates for all positive integers n

$$\mid \mathit{Log}(S_\gamma(w)) - P^n_\gamma(w) \mid \leq \tilde{C}^F_{2n} |\gamma|^{2n} \delta^{-2n-1} (2n)!$$

for all $(w, \gamma) \in W^F_\theta \times U^F_\theta$ *and all* $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *where*

$$\tilde{C}^F_{2n} = \frac{2^4 \left(\frac{2}{\pi}\right)^{2n+1} |\gamma|}{\sqrt{2n-1}(1 - e^{-2\pi \cos\theta})(\mathit{Re}(\gamma)\cos\theta - \mathit{Im}(\gamma)\sin\theta)}.$$

This is similar, but not identical, to the estimates obtained by Garoufalidis and Kashaev, using advanced techniques from the theory of resurgence, in the case where $\mathrm{Re}(\gamma) > 0$, e.g. in the case our $\theta = 0$. Ours is more general, since it does not require that $\mathrm{Re}(\gamma) > 0$, in fact, we see that as we vary $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the above theorem applies to all $\gamma \in \mathbb{C} - \mathbb{R}_-$.

We recall that for all $z \in \mathbb{C} - i\pi\mathbb{Z}$

$$\frac{1}{\sinh(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{z - i\pi n} + \frac{(-1)^n}{z + i\pi n} \right) = \frac{1}{z} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n z}{z^2 + \pi^2 n^2},$$

where the last series converges uniformly on

$$O_\epsilon = \mathbb{C} - \sqcup_{n \in \mathbb{Z} - \{0\}} D(i\pi n, \epsilon)$$

for all $\epsilon > 0$. Thus

$$\varphi_z^F(\gamma) = \frac{1}{\sinh(\gamma z)} - \frac{1}{\gamma z} = \frac{2z}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{\frac{1}{\gamma}}{(\frac{1}{\gamma})^2 + (\frac{z}{\pi n})^2}$$

converges uniformly in

$$\gamma \in O_{\epsilon, z} = \mathbb{C} - \sqcup_{n \in \mathbb{Z} - \{0\}} D(\frac{i\pi n}{z}, \epsilon)$$

for $\epsilon > 0$ and $z \neq 0$.

For

$$\frac{1}{|\gamma|^2}\begin{pmatrix}\mathrm{Re}(\gamma)\\ \mathrm{Im}(\gamma)\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} > \frac{1}{\pi}\left|\begin{pmatrix}\mathrm{Im}(z)\\ \mathrm{Re}(z)\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right|$$

we immidiately check that

$$\mathcal{L}_\theta^{-1}\left(\frac{\frac{1}{\gamma}}{(\frac{1}{\gamma})^2+(\frac{z}{\pi n})^2}\right)(\xi) = \cos(\frac{\xi z}{\pi n}).$$

since

$$\mathcal{L}_\theta(\cos(\frac{\xi z}{\pi n}))(\gamma) = \int_{\mathbb{R}_\theta+} e^{-\xi/\gamma}\cos(\frac{\xi z}{\pi n})d\xi = \frac{1}{2}\left(\frac{1}{\frac{1}{\gamma}+i\frac{z}{\pi n}}+\frac{1}{\frac{1}{\gamma}-i\frac{z}{\pi n}}\right)$$

and so

$$\mathcal{L}_\theta^{-1}(\varphi_z^F)(\xi) = \frac{2z}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(\frac{\xi z}{\pi n}).$$

So we get that

$$B_w(\xi) = \int_{\Gamma_w^F} K_F(w,z)\mathcal{L}^{-1}(\varphi_z^F)(\xi)dz =$$

$$\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\int_{\Gamma_w^{F,\gamma}}\frac{e^{(w+i\frac{\xi}{\pi n})z}}{\sinh(\pi z)}dz + \int_{\Gamma_w^{F,\gamma}}\frac{e^{(w-i\frac{\xi}{\pi n})z}}{\sinh(\pi z)}dz\right)$$

Let

$$V_{w,\theta} = \left\{\xi\in\mathbb{C}\middle|\begin{pmatrix}\pi - \mathrm{Re}(w)\\ \mathrm{Im}(w)\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} > \frac{1}{\pi}\left|\begin{pmatrix}\mathrm{Im}(\xi)\\ \mathrm{Re}(\xi)\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right|\right\}.$$

recall

$$\frac{1}{1+e^{-iw}} = \frac{i}{2}\int_{\Gamma_w^F}\frac{e^{wz}}{\sinh(\pi z)}dz.$$

Theorem 18 (A.)

For all $0 < \delta < \pi \cos\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $w \in W_\theta^F$ *and* $\xi \in V_{w,\theta}$ *we have that*

$$B_w(\xi) = \frac{2}{i\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(\frac{1}{1 + e^{-i(w + i\frac{\xi}{\pi n})}} + \frac{1}{1 + e^{-i(w - i\frac{\xi}{\pi n})}} \right)$$

This formula matches the one obtained by Garoufalidis and Kashaev in the special case where $\theta = 0$, e.g. in the case where $\mathrm{Re}(\gamma) > 0$. From this formula we can obtain the Stokes coefficients by the residue theorem.

We observe that B_w actually extends to a meromorphic function on all of $\mathbb{C}$, e.g. $B_w \in \mathcal{M}(\mathbb{C})$ for all

$$w \in \mathbb{C} - (\pi + 2\pi\mathbb{Z}).$$

Furthermore its poles are

$$P_{B_w} = \pm i\pi\mathbb{Z}_+(\pi - w + 2\pi\mathbb{Z}).$$

Thank you very much for your attention!