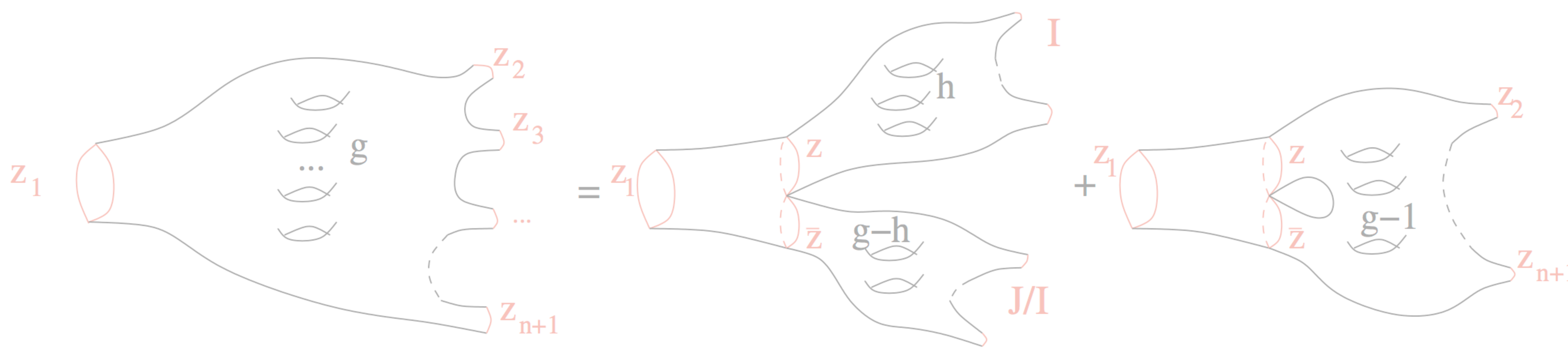
Quantizing curves from Topological Recursion

ReNewQuantum

ERC SYNERGY GRANT

B. Eynard, IPHT, IHES, CRM



Introduction: quantizing curves

WKB and topological recursion

WKB

Quantum curve

Schrödinger equation: $$\left(\hbar^2 \frac{d^2}{dx^2} - \hat{P}(x,\hbar)\right)\psi(x,\hbar) = 0$$

Classical curve

WKB Asymptotic expansion: $\log \psi(x) \sim \sum_{k=-1}^{\infty} \hbar^k S_k(x)$ formal $\in \hbar^{-1}\mathbb{C}[[\hbar]]$

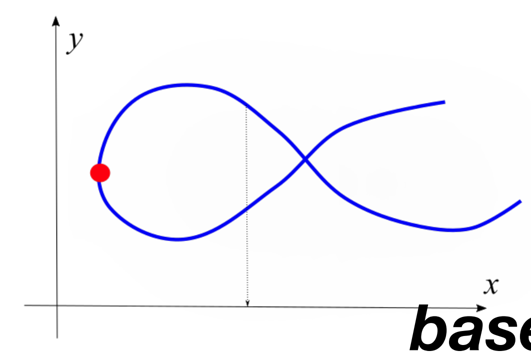
$P(x) = \hat{P}(x,0)$

- **Leading term:** $S_{-1} = \int_*^x y dx$ $\quad dS_{-1}(x)/dx = y$ $\quad dS_{-1}(x) = y dx$ solution of $y^2 - P(x) = 0$

- **subleading terms:** $k > 0$

  $dS_k(x)$ = meromorphic 1-form on the Riemann surface Σ of equation

  $dS_k(x)$ has poles at ramification points



- **Constraint:** to each order in $\hbar$ $\quad \psi(x)$ must be analytic at ramification points

  $\mathrm{Res}_a\, \psi_+(x)\psi_-(x) dx = 0$ in $\mathbb{C}[[\hbar]]$ $\qquad \psi_\pm(x) \to y = \pm\sqrt{P(x)}$

  Hirota

- **Question:** Can we recover all the coefficients, i.e. the differentials $dS_k(x)$ from the geometry of the classical curve ?

How do we quantize a classical curve ? $P(x,y) = 0$ $\qquad y \to \hat{y} = \hbar \frac{d}{dx}$ $\qquad [\hat{y}, x] = \hbar$

$\hat{P}(x,\hat{y},\hbar)\,.\,\psi = 0$

Quantum curve : wrong way

Start from a classical curve $P(x,y)=0$

- **Define** $S_{-1} = \int_*^x y dx$ i.e. $dS_{-1} = y dx$
- **Choose** (*arbitrarily*) meromorphic differentials dS_k with poles at ramification points
- **Define** $\log \psi(x) \sim \sum_{k=-1}^{\infty} \hbar^k S_k(x)$ in $\hbar^{-1}\mathbb{C}[[\hbar]]$
- **Define** $\hat{P}(x,\hbar) = \frac{1}{\psi} \hbar^2 \frac{d^2}{dx^2}\psi$ in $\mathbb{C}[[\hbar]]$
- **Then:** $\left(\hbar^2 \frac{d^2}{dx^2} - \hat{P}(x,\hbar)\right)\psi(x,\hbar) = 0$

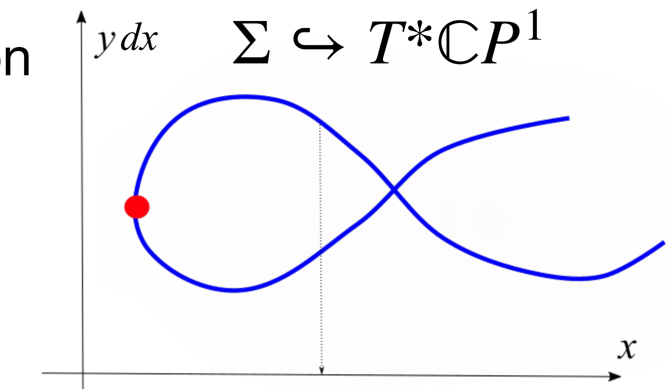
- **Doesn't work:** $\hat{P}(x)$ has poles at ramification points at all orders in $\hbar$

  Constraint: dS_k

Quantum curve from topological recursion

Start from a classical plane curve $P(x,y)=0$ Riemann surface Σ

✦ **Define** $S_{-1} = \int_*^x y dx$ i.e. $dS_{-1} = ydx = \omega_{0,1}$ = Liouville 1-form on $\Sigma \hookrightarrow T^*\mathbb{C}P^1$



✦ **Topological Recursion** —> family of meromorphic multi-differentials $\omega_{g,n}$ with poles at ramification points

Geometry

✦ **Define** $S_k(x) = \sum_{2g-2+n=k} \frac{1}{n!} \overbrace{\int_*^x \dots \int_*^x}^{n} \omega_{g,n}$ ex: $S_1 = \int_*^x \omega_{1,1} + \frac{1}{6} \int_*^x \int_*^x \int_*^x \omega_{0,3}$

✦ **Define** $\log \psi(x) \sim \sum_{k=-1}^{\infty} \hbar^k S_k(x) \in \hbar^{-1}\mathbb{C}[[\hbar]]$ $+ TR.non-pert$ $\in transseries\ \mathbb{C}[[[\hbar]]] \ni e^{\hbar^{-1}}$

✦ **Then look for a polynomial differential operator:** $\hat{P}(x, \hbar \frac{d}{dx}, \hbar)\ \psi(x,\hbar) = 0$

✦ **Works ?** Nearly yes… but…

dS_k is not a meromorphic 1-form except in some good cases Σ genus 0

✦ **Works ?** YES with non-perturbativeT.R. completion = Conjecture… ?

[E 2008, E-Mariño 2008]

Known cases

Conjecture: $\log \psi(x) \sim \sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int \ldots \int \omega_{g,n} \quad + T.R.\ non-pert \quad \in transseries\ \mathbb{C}[[[\hbar]]] \ \ni e^{\hbar^{-1}}$

✦ [Borot-E 2012] Hirota satisfied to $O(\hbar^4)$ for all compact plane curves

✦ **True for Random matrix models**

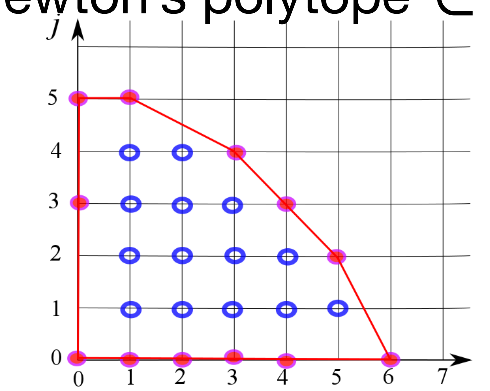
✦ **Airy:** $y^2 - x = 0$ $\quad \left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi = 0 \quad \psi(x) = Ai(\hbar^{-2/3}x)$

[Bergère-E 2008, + others later…]

✦[Bouchard-E 2016] Admissible spectral curves $0 = P(x,y) = \sum_{i,j \in N} P_{i,j} x^i y^j$ N=Newton's polytope $\subset \mathbb{Z}^2$

N with empty interior —> *genus=0*

[Bouchard-ChiDamBaram-Dauphinee 2018]



✦ [Bergère-Borot-Eynard,Marchal,Belliard]

Topological type integrable systems

✦ [E 2017] *Geometry and cycles*

✦ [Iwaki, Marchal, Saenz] *Approach from integrable systems*

✦ [Marchal, Orantin 19,19] *Approach from integrable systems* *Hyperelliptical curves* $y^2 = P(x)$

✦ [E-Garcia-Failde 19] *Mixture of all*

✦ [Dumitrescu-Mulase 2012+]

✦ [Dijkgraaf-Fuji-Manabe, Borot-E] [Borot-Brini]
[Mulase-Sulkowski] [Brini-E-Mariño]

✦ **Many other case by case**

Topological Recursion
and
spectral curves

Spectral curves

T.R.

Spectral curve $\mathcal{S}$ $\longrightarrow$ Invariants $\omega_{g,n}(\mathcal{S})$ $g,n = 0,\dots,\infty$

Def: Spectral curve $\mathcal{S} = [(\Sigma, \pi, \eta, B)]$

✦ $\Sigma =$ smooth surface $\pi : \Sigma \to \Sigma_0 = \mathbb{C}P^1$ Ramified cover of a base $\Sigma_0 =$ Riemann Surface

(not necessarily compact nor connected)

Ramification points (*simple*) $a \in R$

local involution $\pi(\sigma_a(z)) = \pi(z)$

$\pi^*\Sigma_0$ $\to$ Complex structure on Σ

✦ $\eta =$ Meromorphic 1-form on Σ

✦ $B =$ Symmetric meromorphic $1 \otimes 1$ form on $\Sigma \times \Sigma$ $B \in H^0(\Sigma \times \Sigma, K_\Sigma \boxtimes K_\Sigma(2\,\mathrm{diag}_{\Sigma\times\Sigma})^{\mathrm{sym}})_1$

✦ Morphism $(\tilde\Sigma, \phi^*\pi, \phi^*\eta, \phi^*B) \to (\Sigma, \pi, \eta, B)$ $\phi \in C^\infty(\tilde\Sigma, \Sigma)$

Spectral curve $\mathcal{S} = [(\Sigma, \pi, \eta, B)]$ mod isomorphisms $\mathbb{S} =$ {Spectral curves}

Def: Rescaling Spectral curve $\lambda\mathcal{S} = [(\Sigma, \pi, \lambda\eta, B)]$ $\lambda \in \mathbb{C}^*$

Def: $\mathfrak{M}^1 \to \mathbb{S}$ Fiber = meromorphic 1-forms on Σ $\mathfrak{M}^1(\mathcal{S})$ ∞ dimensional vector space

Topological Recursion

Spectral curve $\mathcal{S} = [(\Sigma, \pi, \eta, B)]$ T.R. → Invariants $\omega_{g,n}(\mathcal{S})$

Def: Topological Recursion

$$\omega_{0,1}(\mathcal{S}) = \eta \qquad \omega_{0,2}(\mathcal{S}) = B \qquad K_a(z_1, z) = \frac{\frac{1}{2}\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_1, .\,)}{\omega_{0,1}(z) - \sigma_a^* \omega_{0,1}(z)}$$

$2g - 2 + n > 0 \quad , n \geq 1$

$$\omega_{g,n}(\mathcal{S}; z_1, \ldots, z_n) = \sum_{a \in R} \operatorname*{Res}_{z \to a} K_a(z_1, z) \Big[\omega_{g-1,n+1}(\mathcal{S}; z, \sigma_a(z), z_2, \ldots, z_n) + \sum_{g_1+g_2=g,\ I_1 \sqcup I_2 = \{z_2,\ldots,z_n\}}^{no\ (0,1)} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(\sigma_a(z), I_2) \Big]$$

$\omega_{g,n}(\mathcal{S}) \in H^0(\Sigma^n, K_\Sigma^{\boxtimes n}(*R)^{\mathfrak{S}_n}) \qquad \omega_{g,0}(\mathcal{S}) = F_g(\mathcal{S}) \in \mathbb{C}$

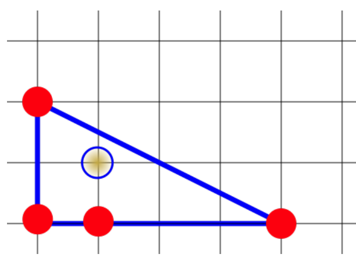
Operator notation: $K : \mathfrak{M}^1(\mathcal{S}) \otimes \mathfrak{M}^1(\mathcal{S}) \to \mathfrak{M}^1(\mathcal{S})$

$$\omega \otimes \tilde{\omega} \mapsto \sum_{a \in R} \operatorname*{Res}_{z \to a} K_a(z_1, z)\ \omega(z)\ \tilde{\omega}(\sigma_a(z))$$

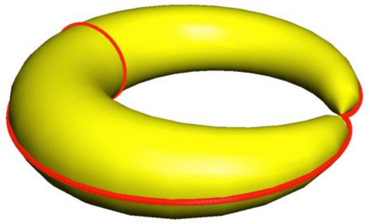
$$\omega_{g,n}(\mathcal{S}; z_1, \ldots, z_n) = K(\omega_{g-1,n+1}) + \sum_{g_1+g_2=g,\ I_1 \sqcup I_2 = \{z_2,\ldots,z_n\}}^{\text{no } (0,1)} K\Big(\omega_{g_1,1+|I_1|}(\,.\,, I_1) \otimes \omega_{g_2,1+|I_2|}(\,.\,, I_2)\Big)$$

Painlevé 1

Classical curve



$y^2 = x^3 + tx + 2u_0^3$



time $\quad t = -3u_0^2$

$$\mathcal{S}_t \begin{cases} \Sigma = \mathbb{C} \\ x = \pi(z) = z^2 - 2u_0 \\ y = z^3 - 3u_0 z \end{cases} \qquad B(z_1,z_2) = \frac{1}{(z_1-z_2)^2} dz_1 \otimes dz_2 \qquad ydx = \eta(z) = \left(z^3 - 3u_0 z\right) 2z dz$$

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{d_1,\ldots,d_n} \frac{6^{2-2g-n} u_0^{5-5g-2n}}{(3g-3+n-\sum_i d_i)!} \left\langle \tau_2^{3g-3+n-\sum_i d_i} \tau_{d_1}\ldots\tau_{d_n} \right\rangle_g \prod_{i=1}^n \frac{u_0^{d_i}(2d_i+1)!!\, dz_i}{z_i^{2d_i+2}}$$

$\in \mathbb{Q}$ Kontsevich-Witten Intersection numbers

$$n=0: \quad F_g = \omega_{g,0} = u_0^{5-5g} \frac{6^{2-2g}}{(3g-3)!} \left\langle \tau_2^{3g-3} \right\rangle_g = (-t/3)^{\frac{5-5g}{2}} \frac{6^{2-2g}}{(3g-3)!} \left\langle \tau_2^{3g-3} \right\rangle_g$$

$$\text{ex}: \left\langle \tau_2^3 \right\rangle_2 = \frac{7}{240}$$

2nd derivative $\quad u_g = \dfrac{d^2 F_g}{dt^2} = (-t)^{\frac{1-5g}{2}} (5-5g)(3-5g) \dfrac{2^{-2g}\, 3^{\frac{g-1}{2}}}{(3g-3)!} \left\langle \tau_2^{3g-3} \right\rangle_g$

$$u = u_0 + \frac{\hbar^2}{48t^2} + \sum_{g=2}^{\infty} \hbar^{2g} u_g$$

Satisfies **Painlevé 1 équation** $\quad \dfrac{\hbar^2}{2} \dfrac{d^2}{dt^2} u + 3u^2 = -t$

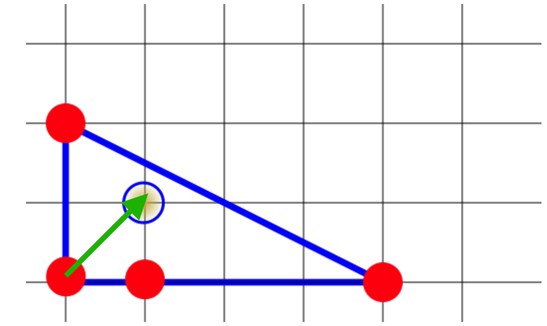
[Bergère-Borot-Eynard,Marchal,Belliard] [E book 2016]

Associated ODE $\quad \psi(x) \sim e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int\ldots\int \omega_{g,n}} \qquad \tilde\psi(x) \sim e^{\sum_{g,n} \frac{(-1)^n \hbar^{2g-2+n}}{n!} \int\ldots\int \omega_{g,n}}$

Quantum curve

$$\hbar \frac{d}{dx} \begin{pmatrix} \psi \\ \tilde\psi \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2}\frac{du}{dt} & x-u \\ (x+2u)(x-u) + \frac{\hbar^2}{4}\frac{d^2u}{dt^2} & -\frac{\hbar}{2}\frac{du}{dt} \end{pmatrix} \begin{pmatrix} \psi \\ \tilde\psi \end{pmatrix} \qquad \hbar \frac{d}{dt} \begin{pmatrix} \psi \\ \tilde\psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x+2u & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \tilde\psi \end{pmatrix}$$

Painlevé 1



time $t = -3u_0^2$ **Classical curve** $y^2 = x^3 + tx + 2u_0^3$

$$\mathcal{S}_t \begin{cases} \Sigma = \mathbb{C} \\ x = z^2 - 2u_0 \\ y = z^3 - 3u_0 z \end{cases} \qquad B(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} dz_1 \otimes dz_2$$

Classical curve

Associated ODE $\psi(x) \sim e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int \dots \int \omega_{g,n}}$

$$\hbar \frac{d}{dx} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = \mathcal{D}(x) \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} \qquad \mathcal{D}(x) = \begin{pmatrix} \frac{\hbar}{2} \frac{du}{dt} & x - u \\ (x + 2u)(x - u) + \frac{\hbar^2}{4} \frac{d^2u}{dt^2} & -\frac{\hbar}{2} \frac{du}{dt} \end{pmatrix}$$

$$\hbar \frac{d}{dt} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x + 2u & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}$$

« Quantum curve »

$$\det(y - \mathcal{D}(x)) = y^2 - x^3 - tx - 2u_0^3 - \hat{V}$$

Quantum curve

Interior point (after shift ↗)

$$\hat{V} = O(\hbar^2) \in \hbar^2 \mathbb{C}[[\hbar]]$$

Theorem: Loop equation

TR $\Longrightarrow$ $\omega_1^2(x) + \omega_2(x, x) = (\ x^3 + tx + 2u_0^3 + \tilde{V}\ ) . dx^2$ $\qquad \omega_n = \sum_g \hbar^{2g-2+n} \omega_{g,n}$

[E-Garcia-Failde 19] $\tilde{V} = \hbar^2 \frac{dF}{dt} = \sum_{g=1}^{\infty} \hbar^{2g} \frac{dF_g}{dt}$

!!! fact : $\hat{V} = \tilde{V}$ **Quantum curve polynomial = loop equation polynomial**

Loop equations

(here: hyperelliptical curve version) $y^2 = P(x)$

Theorem : [CEO 2005, EO 2007] **TR —>**

$$P_{g,n}(x; z_1, \ldots, z_n) dx^2 := \omega_{g-1,n+2}(z, z, z_1, \ldots, z_n) + \sum_{h+h'=g} \sum_{I+I'=\{z_1,\ldots,z_n\}} \omega_{h,1+|I|}(z, I)\omega_{h',1+|I'|}(z, I')$$

has no pole at ramification points $x = \pi(z),\ x_j = \pi(z_j)$

Corollary : *(here: hyperelliptical curve version)*

$$\tilde{P}_{g,n}(x; z_1, \ldots, z_n) = P_{g,n}(x; z_1, \ldots, z_n) - \sum_{j=1}^{n} \frac{d}{dx_j} \frac{\omega_{g,n}(z, z_1, \hat{z}_j, \ldots, z_n) - \omega_{g,n}(z_1, \ldots, z_n)}{x - x_j}$$

Has no pole at ramification points , and no pole at coinciding points

Its only poles can come from $\omega_{0,1} = y dx$

$$\tilde{P}_{0,0}(x) = y^2 = P(x)$$

$$\tilde{P}_{g,n}(x; z_1, \ldots, z_n) = \sum_k x^k\ \tilde{P}_{g,n;k}(z_1, \ldots, z_n) = 2y\omega_{g,n+1}(z, z_1, \ldots, z_n) + \text{analytic at } \infty$$

$$\tilde{P}_{g,n;k}(z_1, \ldots, z_n) = 2 \operatorname{Res} x^{-k-1}\sqrt{P(x)}\ \omega_{g,n+1}(z, z_1, \ldots, z_n)$$

Theorem [EO2007]:

$$t_k = \operatorname{Res} x^{-k/2} y dx \qquad \frac{\partial}{\partial t_k}\omega_{g,n}(z_1, \ldots, z_n) = \frac{1}{k} \operatorname{Res} x^{k/2}\omega_{g,n+1}(z, z_1, \ldots, z_n)$$

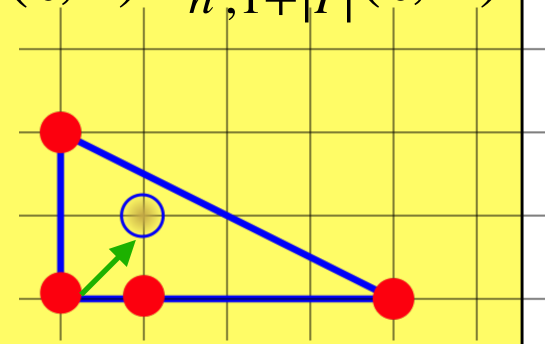
Loop equations

(here: hyperelliptical curve version) $y^2 = P(x)$

Theorem :

$$\tilde{P}_{g,n}(x; z_1, \ldots, z_n) dx^2 := \omega_{g-1,n+2}(z, z, z_1, \ldots, z_n) + \sum_{h+h'=g} \sum_{I+I'=\{z_1,\ldots,z_n\}} \omega_{h,1+|I|}(z, I)\omega_{h',1+|I'|}(z, I')$$

$$- \sum_{j=1}^{n} \frac{d}{dx_j} \frac{\omega_{g,n}(z, z_1, \hat{z}_j, \ldots, z_n) - \omega_{g,n}(z_1, \ldots, z_n)}{x - x_j}$$



is given by

$$\tilde{P}_{g,n}(x; z_1, \ldots, z_n) = L(x, t)\ \omega_{g,n}(z_1, \ldots, z_n) \qquad L(x, t) = \sum_{j,k} t_{k+1} x^{k-1-j} \frac{\partial}{\partial t_j}$$

$t_k = \operatorname{Res} x^{-k/2} y dx \qquad x = \pi(z),\ x_j = \pi(z_j)$

Interior points (after shift ↗)

[E-GF 2019]
[O-M 2019]

Integrate $z_1, \ldots, z_n$

$$F_{g,n}(x, \tilde{x}) = \int_{\tilde{x}}^{x} \cdots \int_{\tilde{x}}^{x} \omega_{g,n} \qquad F'_{g,n}(z; x, \tilde{x}) = \int_{\tilde{x}}^{x} \cdots \int_{\tilde{x}}^{x} \omega_{g,n+1}(z, .) \qquad F''_{g,n}(z, z'; \tilde{x}, x) = \int_{\tilde{x}}^{x} \cdots \int_{\tilde{x}}^{x} \omega_{g,n+2}(z, z', .)$$

$$\int_{\tilde{x}}^{x} \cdots \int_{\tilde{x}}^{x} \tilde{P}_{g,n}(x; z_1, \ldots, z_n) = L(x, t)\ F_{g,n}(x, \tilde{x})$$

See backup divisors

$$\int_{\tilde{x}}^{x} \cdots \int_{\tilde{x}}^{x} \tilde{P}_{g,n}(x; z_1, \ldots, z_n) = F''_{g-1,n+2}(z, z, x, \tilde{x}) + \sum_{h+h'=g} \sum_{m+m'=n} \frac{n!}{m!m'!} F'_{h,m+1}(z, x, \tilde{x}) F'_{h',m'+1}(z, x, \tilde{x}) - n \sum_{j} \alpha_j \frac{F'_{g,n}(z; x, \tilde{x}) - F'_{g,n}(z_j;}{x - x_j}$$

Multiply by $\frac{\hbar^{2g-2+n}}{n!}$ and sum over g, n

[Bergère-E 2008,
Bouchard E 2016]
[E-GF 2019, O-M 2019]

—> linear PDE

$$\hbar^2 \frac{d^2}{dx^2} \psi(x, \tilde{x}) + \hbar \frac{d\psi(x, \tilde{x})/dx - d\psi(x, \tilde{x})/d\tilde{x}}{x - \tilde{x}} = P(x)\psi(x, \tilde{x}) + \hbar L(x, t) . \psi(x, \tilde{x})$$

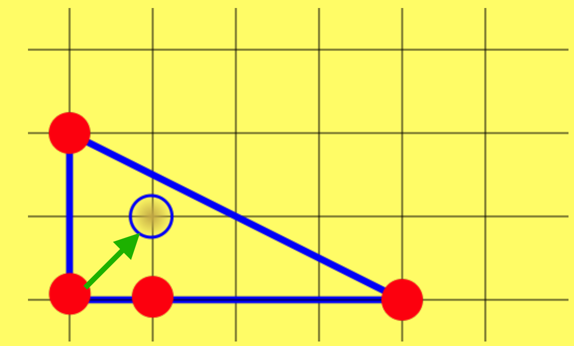
Loop equations process

(here: hyperelliptical curve version) $y^2 = P(x)$

Theorem : *n*=0

$$\tilde{P}_{g,0}(x)dx^2 := \omega_{g-1,2}(z,z) + \sum_{h+h'=g} \omega_{h,1}(z)\omega_{h',1}(z)$$

is a polynomial



$$\tilde{P}_{g,0}(x) = \sum_k x^k \tilde{P}_{g,0;k} \qquad (= L(x,t)F_g)$$

Interior points (after shift ↗)

Def : $$\tilde{P}(x) = P(x) + \sum_{g=1}^{\infty} \hbar^{2g} \tilde{P}_{g,0}(x) \quad = P(x) + O(\hbar^2)$$

Process : start from a polynomial $P(x)$ → define a new polynomial $\tilde{P}(x) = P(x) + O(\hbar^2)$

Beyond hyperelliptical generalisation :

start from a classical curve $P(x,y)$ → define a new curve $\tilde{P}(x,y) = P(x,y) + O(\hbar^2, \text{interior})$

Fixed point

Iterate

$\mathcal{S} : P(x,y) = 0$ → $\tilde{\mathcal{S}} : \tilde{P}(x,y) = \sum_{g=0}^{\infty} \hbar^{2g} \tilde{P}_g(x,y)$

Fixed point: (if it exists)

$\tilde{\mathcal{S}} = \mathcal{S}$ $\tilde{P} = P$

Remark:

this is necessarily the case if no interior points *[Bouchard-E 2016]* Admissible spectral curves

Remark:

For all matrix models $\exists$ a deformation $\hat{\mathcal{S}} = \mathcal{S} + O([[[\hbar]]])$ which is a fixed point $\tilde{\hat{\mathcal{S}}} = \hat{\mathcal{S}}$

Does a fixed point always exist ?

Deformations and cycles

[E 2017]

Example KdV

« Generalized cycle » $\mathfrak{M}_1(\mathcal{S})$ Painlevé 1: $t_1 \neq 0$

Example **KdV** $\mathcal{S}$

$$\Sigma = \mathbb{C} \qquad \pi(z) = z^2 \qquad \eta(z) = 2z^2\left(1 + \frac{1}{2}\sum_{k=1}^{\infty} t_k z^{2k}\right) dz \qquad B(z,z') = \frac{1}{(z-z')^2} dz \otimes dz'$$

Def: $\mathcal{B}_{\infty,k}$ $k \geq 1$

$\mathcal{B}_{\infty,k} \in \mathfrak{M}^1(\mathcal{S})^*$ defined by $< \mathcal{B}_{\infty,k}, \nu > := \frac{1}{k} \operatorname{Res}_{z' \to \infty} z'^k \nu(z')$ $\forall \nu \in \mathfrak{M}^1(\mathcal{S})$

« Generalized cycle »

$$< \mathcal{B}_{\infty,k}, B > = \frac{1}{k} \operatorname{Res}_{z' \to \infty} z'^k B(z,z') = -z^{k-1} dz \implies < \mathcal{B}_{\infty,k}, B > \in \mathfrak{M}^1(\mathcal{S}) \qquad \mathcal{B}_{\infty,k} \in \mathfrak{M}_1(\mathcal{S})$$

Def: of the cycle $\mathcal{A}_{\infty,k}$ $k \geq 0$

$\mathcal{A}_{\infty,k} \in \mathfrak{M}^1(\mathcal{S})^*$ defined by $< \mathcal{A}_{\infty,k}, \nu > := 2\pi i \operatorname{Res}_{z' \to \infty} z'^{-k} \nu(z')$

« Generalized cycle »

$$< \mathcal{A}_{\infty,k}, B > = 2\pi i \operatorname{Res}_{z' \to \infty} z'^{-k} B(z,z') = 0 \qquad < \mathcal{A}_{\infty,k}, B > = 0 \in \mathfrak{M}^1(\mathcal{S}) \implies \mathcal{A}_{\infty,k} \in \mathfrak{M}_1(\mathcal{S})$$

Def: Generalized cycles $\mathfrak{M}_1(\mathcal{S}) = \{\gamma \in \mathfrak{M}^1(\mathcal{S})^* \mid < \gamma, B > \in \mathfrak{M}^1(\mathcal{S})\}$ *[Eynard 2017]*

Symplectic space $\gamma \cap \gamma' := \frac{1}{2\pi i}\left(\int_\gamma \int_{\gamma'} B - \int_{\gamma'} \int_\gamma B\right)$ pairing = « integration » $\int_\gamma \omega := < \gamma, \omega >$

$$\mathcal{A}_{\infty,k} \cap \mathcal{B}_{\infty,l} = \delta_{k,l}$$

Example KdV

Example **KdV** $\mathcal{S}$

$$\Sigma = \mathbb{C} \qquad \pi(z) = z^2 \qquad \eta(z) = \omega_{0,1}(z) = 2z^2 \left(1 + \frac{1}{2}\sum_{k=1}^{\infty} t_k z^{2k}\right) dz \qquad B(z,z') = \frac{1}{(z-z')^2} dz \otimes dz'$$

Remark: $t_k = \frac{1}{2\pi i} \int_{\mathcal{A}_{\infty,2k+3}} \omega_{0,1}$ $\qquad \frac{\partial}{\partial t_k} \omega_{0,1}(z) = z^{2k+2} dz = \int_{\mathcal{B}_{\infty,2k+3}} B = \int_{\mathcal{B}_{\infty,2k+3}} \omega_{0,2}$

$$\omega_{g,n}(z_1,\ldots,z_n) = (-1)^n \, 2^{2-2g-n} \sum_{d_1,\ldots,d_n} \left\langle e^{\frac{1}{2}\sum_k (2k+1)!! t_k \tau_{k+1}} \tau_{d_1} \ldots \tau_{d_n} \right\rangle_g \prod_{i=1}^{n} \frac{(2d_i+1)!! \, dz_i}{z_i^{2d_i+2}}$$

$$\frac{\partial}{\partial t_k} \omega_{g,n}(z_1,\ldots,z_n) = \frac{1}{2k+3} \operatorname*{Res}_{z'\to\infty} z'^{2k+3} \omega_{g,n+1}(z_1,\ldots,z_n,z') = \int_{\mathcal{B}_{\infty,2k+3}} \omega_{g,n+1}(z_1,\ldots,z_n,.)$$

Theorem : Deformations *[EO 2007]*

$$t_\gamma = \frac{1}{2\pi i} \int_{\gamma^\perp} \eta \qquad \frac{\partial}{\partial t_\gamma} \omega_{g,n} = \int_\gamma \omega_{g,n+1}$$

Deformations

Deformations: Tangent Space $T_{\mathcal{S}}\mathbb{S}$ ⟷ Space of cycles $\mathfrak{M}_1(\mathcal{S})$

Def : **map: cycles -> deformations** $\partial : \mathfrak{M}_1(\mathcal{S}) \to t_{\mathcal{S}}\mathbb{S}$ **Theorem : surjective**

[Eynard 2017]

$$\partial_\gamma \omega_{g,n} = \int_\gamma \omega_{g,n+1}$$

$$\partial_\gamma \omega_{g,n}(\mathcal{S}; z_1, \ldots, z_n) = \int_{z' \in \gamma} \omega_{g,n+1}(\mathcal{S}; z_1, \ldots, z_n, z')$$

Def : **Finite deformation (follow tangent field):** $e^{t\partial_\gamma}$

$$e^{t\partial_\gamma}\mathcal{S} = \mathcal{S} + t\gamma$$

$$\omega_{g,n}(\mathcal{S} + t\gamma; z_1, \ldots, z_n) = \sum_m \frac{t^m}{m!} \int_{z_1' \in \gamma} \cdots \int_{z_m' \in \gamma} \omega_{g,n+m}(\mathcal{S}; z_1, \ldots, z_n, z_1', \ldots, z_m')$$

$$F_g(\mathcal{S} + t\gamma) = \omega_{g,0}(\mathcal{S} + t\gamma) = \sum_m \frac{t^m}{m!} \int_{z_1' \in \gamma} \cdots \int_{z_m' \in \gamma} \omega_{g,m}(\mathcal{S}; z_1', \ldots, z_m')$$

Remark: $\psi(\mathcal{S}; z) = \dfrac{1}{Z(\hbar^{-1}\mathcal{S})}\, e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_*^z \cdots \int_*^z \omega_{g,n}}$

$$\gamma_D = \text{chain } * \to z$$

$$\partial\gamma_D = [z] - [\,*\,]$$

$$Z(\hbar^{-1}\mathcal{S}) = e^{\sum_g \hbar^{2g-2} F_g(\mathcal{S})}$$

$$\psi(\mathcal{S}; x) = \frac{1}{Z(\hbar^{-1}\mathcal{S})}\, Z(\hbar^{-1}\mathcal{S} + \gamma_D)$$

Def : Insertion operator Special choice of « cycle »: $\gamma = \mathrm{ev}_z$

$$\Delta_z = \partial_{\mathrm{ev}_z}$$

$$\Delta\omega_{g,n} = \omega_{g,n+1}$$

$$\Delta_z \omega_{g,n}(\mathcal{S}; z_1, \ldots, z_n) = \omega_{g,n+1}(\mathcal{S}; z_1, \ldots, z_n, z)$$

Remark : $\partial_\gamma = \int_\gamma \Delta$

Loop equations

Start with an algebraic spectral curve $\mathcal{S} : P(x,y) = 0$

- **Case hyperelliptic curve:** 2 preimages of $x = \{z_+(x), z_-(x)\}$

$$\tilde{P}_{g,0}(x) := \omega_{g-1,2}(z_+, z_-) + \sum_{h=0}^{g} \omega_{h,1}(z_+)\omega_{g-h,1}(z_-) = \Delta_{z_+(x)}\Delta_{z_-(x)}F_{g-1} + \sum_{h=0}^{g} \Delta_{z_+(x)}F_h\Delta_{z_-(x)}F_{g-h}$$

- **General case, rank *r* :** r preimages of $x = \{z_1(x), \ldots, z_r(x)\}$

$$\tilde{P}_{g,0}(x) := \sum_{1\le i<j\le r} \Delta_{z_i(x)}\Delta_{z_j(x)}F_{g-1} + \sum_{h=0}^{g} \Delta_{z_i(x)}F_h\Delta_{z_j(x)}F_{g-h}$$

$$\tilde{P}(x,y) = \frac{1}{Z(\hbar^{-1}\mathcal{S})}\prod_{i=1}^{r}(y - \Delta_{z_i(x)})\ Z(\hbar^{-1}\mathcal{S})$$

$$Z(\hbar^{-1}\mathcal{S}) = e^{\sum_g \hbar^{2g-2}F_g(\mathcal{S})}$$

Theorem (Loop equations):

TR —> $\tilde{P}(x,y)$ is meromorphic on the base, and has no pole at ramification points

$$\tilde{P}(x,y) = (y - \Delta_{z_1(x)})(y - \Delta_{z_2(x)})\ldots(y - \Delta_{z_r(x)})\ e^{\sum_g \hbar^{2g-2}F_g(\mathcal{S})}$$

Process : $\mathcal{S} : P(x,y)$ ⟶ $\tilde{\mathcal{S}} :\quad \tilde{P}(x,y) = P(x,y) + \text{interior points} \ \in \mathbb{C}[[\hbar]]$

$$\tilde{\mathcal{S}} = \mathcal{S} + \hbar^2\tilde{\Gamma} = e^{\hbar^2\partial_{\tilde{\Gamma}}}\mathcal{S} \qquad \tilde{\Gamma} \in H_1(\mathcal{S}, \mathbb{C})[[\hbar]]$$

Summing over the lattice

Start with an algebraic spectral curve $\mathcal{S} : P(x,y) = 0$ $\quad P(\mathcal{S}; x, y) = 0$

Define : $\mathcal{T}(\hbar^{-1}\mathcal{S}) := \sum_{\gamma \in H_1(\Sigma,\mathbb{Z})} Z(\hbar^{-1}\mathcal{S} + \gamma) \quad \in \mathbb{C}[[[\hbar]]]$

TR non perturbative formal trans-series
[E, E-Mariño 2008]

See theta

$$\psi(\hbar^{-1}\mathcal{S}, D) := \frac{1}{\mathcal{T}(\hbar^{-1}\mathcal{S})} \sum_{\gamma \in H_1(\Sigma,\mathbb{Z})} Z(\hbar^{-1}\mathcal{S} + \gamma_D + \gamma)$$

$$\tilde{\mathcal{P}}(\mathcal{S}; x, y) := \frac{1}{\mathcal{T}(\hbar^{-1}\mathcal{S})} \sum_{\gamma \in H_1(\Sigma,\mathbb{Z})} \tilde{P}(\mathcal{S} + \gamma; x, y) \quad = P(x,y) + O([[[\hbar]]])$$

interior points

$$\tilde{\mathcal{P}}(\mathcal{S}; x, y) = P(\mathcal{S} + \tilde{\Gamma}; x, y) \qquad \tilde{\Gamma} \in H_1(\mathcal{S}, \mathbb{C})/H_1(\mathcal{S}, \mathbb{Z})[[[\hbar]]]$$

Iterate $\mathcal{S} \longrightarrow \tilde{\mathcal{S}} = \mathcal{S} + \tilde{\Gamma} \qquad \tilde{\Gamma} \in \mathcal{L} \,/\, H_1(\mathcal{S}, \mathbb{Z}) \qquad \mathcal{L} \subset H_1(\mathcal{S}, \mathbb{C})$

Fixed point: $\hat{\mathcal{S}} = \mathcal{S} + O([[[\hbar]]])$ exists because $\mathcal{L} \,/\, H_1(\mathcal{S}, \mathbb{Z})$ is compact

$$\tilde{\hat{\mathcal{S}}} = \hat{\mathcal{S}} \qquad \tilde{\hat{\mathcal{P}}} = \hat{\mathcal{P}} \qquad \tilde{\hat{\Gamma}} = 0$$

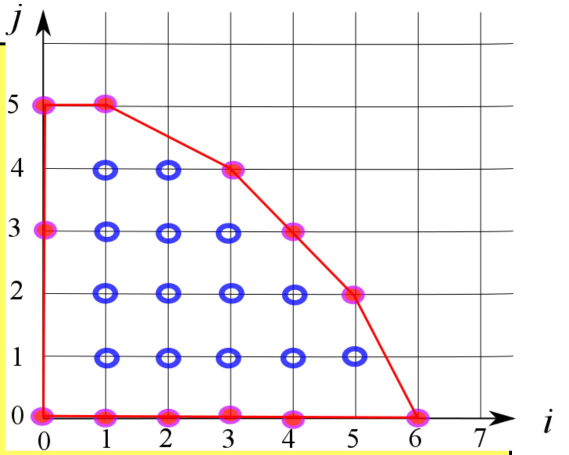
Then follow the same steps $\hbar^2 \frac{d^2}{dx_1^2}\psi(D) + \hbar \sum_{k \neq i} \frac{\alpha_1 d\psi(D)/dx_1 + \alpha_k d\psi(D)/dx_k}{x_1 - x_k} = \hat{\mathcal{P}}(x_1)\psi(D)$

—> linear ODE for $\psi(D) = \psi(\hbar^{-1}\mathcal{S} + \hat{\Gamma}, D)$ —> quantum curve

[E 2017]

Conclusion and prospects

Conclusion



Summary :

- **Classical** curve $\mathcal{S}$ $\quad P(x,y)=0$
- TR $\quad \omega_{g,n}(\mathcal{S}) \quad \psi_{\text{pert}}(\hbar^{-1}\mathcal{S};D) = e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int \dots \int \omega_{g,n}(\mathcal{S})}$

  trans-series $\quad \psi_{\text{non-pert}}(\hbar^{-1}\mathcal{S};D) = \sum_{\gamma \in H_1(\Sigma,\mathbb{Z})} \psi_{\text{pert}}(\hbar^{-1}\mathcal{S} + \hat{\Gamma} + \gamma, D)$

  interior

- Satisfies a **quantum** curve $\quad \hat{P}(x,\hbar d/dx)\,.\,\psi = 0$ whose symbol $\hat{P}(x,y) = P(x,y) + O([[[\hbar]]])$

  where $\hat{P}(x,y)$ is a fixed point of the loop-equations process

- It automatically satisfies $\quad L(x,y,t)\,.\,\psi = 0 \quad$ (derivatives wrt times)

Conclusion : Non-perturbative ***TR*** quantizes the curve *[Eynard arxiv 1706.04938]*

- All this easier to understand in the space of generalized cycles

—> Do geometry !!!
—> Do generalized cycles

Further Prospects

- Resummation of transseries, resurgence
- Wall crossing and Stokes phenomenon, Stokes coefficients
- Non algebraic cases: Example: $P(e^x,e^y)=0$ $\quad \tilde{P}$ not polynomial

  $H_1(\mathcal{S},\mathbb{C})$ / $H_1(\mathcal{S},\mathbb{Z})$ non compact

Thank you for your attention

Back-up

Wave function and divisors

ψ depends on a starting point of integration *

$$\psi(x) = e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_*^x \dots \int_*^x \omega_{g,n}}$$

$$\psi(x, \tilde{x}) = e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_{\tilde{x}}^x \dots \int_{\tilde{x}}^x \omega_{g,n}}$$

Def : Integrating with a divisor

$$D = \sum_{i=1}^{k} \alpha_i [x_i] \qquad \deg D = \sum_{i=1}^{k} \alpha_i = 0$$

$$\int_D := \sum_i \alpha_i \int_*^{x_i} = \int_{\gamma_D} \quad \text{where} \quad \partial\gamma_D = D$$

$$\psi(D) = e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_D \dots \int_D \omega_{g,n}}$$

abuse of language $\psi(\gamma_D) = e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_{\gamma_D} \dots \int_{\gamma_D} \omega_{g,n}}$

Usual: $D = [x] - [\infty]$ $\qquad \psi(D) = \psi(x)$

Back

Gaussian Matrix model

✦ **Example :** Gaussian measure $e^{-\frac{N}{2}\mathrm{Tr}\,M^2}dM$ on H_N $N=\hbar^{-1}$

$\psi(x) := \mathbb{E}(\det(x-M))$

$\psi(x) = H_N(\hbar^{-1/2}x)$ = Hermite polynomial *[Heine, Jacobi]* *[Mehta, Dyson, Wick, Tutte…]*

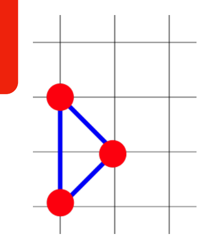
Satisfies a linear ODE

$$\left(\hbar^2\frac{d^2}{dx^2} - x\hbar\frac{d}{dx} + 1\right)\psi = 0$$

Quantum curve

$$y^2 - xy + 1 = 0$$

Classical curve



✦ **Loop equations :** $\omega_1(x) := \mathbb{E}\left(\mathrm{Tr}\,\frac{dx}{x-M}\right)$ « Resolvent »

$$\hat{W}_2(x_1,x_2) := \mathbb{E}\left(\mathrm{Tr}\,\frac{dx_1}{x_1-M}\,\mathrm{Tr}\,\frac{dx_2}{x_2-M}\right)$$

« cumulant »

$$\omega_2(x_1,x_2) := \hat{W}_2(x_1,x_2) - \omega_1(x_1)\omega_1(x_2)$$

Theorem (loop equation) : $\omega_2(x,x) + \omega_1(x)^2 - x\omega_1(x) = -\,dx^2$

✦ **Asymptotic expansions :** $\omega_1(x) \sim \sum_{g=0}^{\infty} N^{1-2g}\omega_{g,1}(x)$ $\omega_{0,1} = y\,dx$ $y^2 - xy + 1 = 0$

Theorem (TR) : $\omega_{g,n} = T.R.$

$$\omega_n(x_1,\dots,x_n) \sim \sum_{g=0}^{\infty} N^{2-2g-n}\omega_{g,n}(x_1,\dots,x_n)$$

✦ **The conjecture :**

$$\psi(x) = \mathbb{E}(e^{\mathrm{Tr}\log(x-M)}) = \mathbb{E}(e^{\mathrm{Tr}\int_*^x \frac{d\tilde{x}}{\tilde{x}-M}}) = \mathbb{E}\left(\sum_n \frac{1}{n!}\int_*^x \mathrm{Tr}\,\frac{d\tilde{x}_1}{\tilde{x}_1-M}\int_*^x \mathrm{Tr}\,\frac{d\tilde{x}_2}{\tilde{x}_2-M}\dots\int_*^x \mathrm{Tr}\,\frac{d\tilde{x}_n}{\tilde{x}_n-M}\right)$$

$$= \sum_n \frac{1}{n!}\int_*^x \dots \int_*^x \hat{W}_n(\tilde{x}_1,\dots,\tilde{x}_n) = e^{\sum_n \frac{1}{n!}\int_*^x\dots\int_*^x \omega_n} = e^{\sum_{g,n}\frac{\hbar^{2g-2+n}}{n!}\int_*^x\dots\int_*^x \omega_{g,n}}$$

Back

Matrix models

Random matrix of size $N = \hbar^{-1}$ measure $e^{-N\,\mathrm{Tr}\,V(M)}dM$

$$\psi(x) := \mathbb{E}(\det(x-M))\; e^{-\frac{N}{2}V(x)}$$

$$\psi(x) = \mathbb{E}(e^{\mathrm{Tr}\log(x-M)}) = \mathbb{E}(e^{\mathrm{Tr}\int_*^x \frac{d\tilde{x}}{\tilde{x}-M}}) = \mathbb{E}\left(\sum_n \frac{1}{n!}\int_*^x \mathrm{Tr}\,\frac{d\tilde{x}_1}{\tilde{x}_1-M}\int_*^x \mathrm{Tr}\,\frac{d\tilde{x}_2}{\tilde{x}_2-M}\dots\int_*^x \mathrm{Tr}\,\frac{d\tilde{x}_n}{\tilde{x}_n-M}\right)$$

$$= \sum_n \frac{1}{n!}\int_*^x \dots \int_*^x \hat{W}_n(\tilde{x}_1,\dots,\tilde{x}_n) = e^{\sum_n \frac{1}{n!}\int_*^x\dots\int_*^x \omega_n} = e^{\sum_{g,n}\frac{\hbar^{2g-2+n}}{n!}\int_*^x\dots\int_*^x \omega_{g,n}}$$

Resolvent $\omega_1(x) = \mathbb{E}(\mathrm{Tr}(x-M)^{-1})dx$ $\qquad \omega_1(x) = \sum_g \hbar^{2g-1}\omega_{g,1}(x) \quad +non-pert \quad (e^{-A/\hbar})$

$$\omega_2(x_1,x_2) = \mathbb{E}\left(\mathrm{Tr}(x_1-M)^{-1}\mathrm{Tr}(x_2-M)^{-1}\right)_c dx_1dx_2$$

$$= \mathbb{E}\left(\mathrm{Tr}(x_1-M)^{-1}\mathrm{Tr}(x_2-M)^{-1}\right)dx_1dx_2 - \omega_1(x_1)\omega_1(x_2)$$

$$\omega_2(x_1,x_2) = \sum_g \hbar^{2g}\;\omega_{g,2}(x_1,x_2) \quad +non-pert$$

Spectral curve $\omega_{0,1} - \frac{1}{2}V'(x)dx = ydx$ $\qquad y^2 = P(x)$ = polynomial

Loop equations $(\omega_1(x) - \frac{1}{2}V'(x)dx)^2 + \omega_2(x,x) = \tilde{P}(x)dx^2$ = polynomial $\qquad \tilde{P}(x) \in \mathbb{C}[[[\hbar]]]$

Linear ODE for $\psi(x)$: $\left(\hbar^2\frac{d^2}{dx^2} - \hat{P}(x)\right).\psi(x) = 0$ $\qquad \hat{P}(x) \in \mathbb{C}[[[\hbar]]]$

Fact : in all random matrix models: $\hat{P}(x) = \tilde{P}(x) \quad = P + O[[[\hbar]]]$

Back

Cycles
generalized cycles

Cycles

Poincaré duality $\gamma =$ cycle, $\omega =$ holomorphic 1-form $\quad < \gamma, \omega > = \int_\gamma \omega \quad \to \quad \gamma \in \Omega^1(\Sigma)^*$

Integrate B : $< \gamma, B > = \Omega_\gamma \qquad \Omega_\gamma(z) = \int_{z' \in \gamma} B(z, z') \qquad \in \Omega^1(\Sigma) \subset \mathfrak{M}^1(\mathcal{S})$

Generalize to $\gamma \in \mathfrak{M}^1(\mathcal{S})^* \qquad < \gamma, B > = \Omega_\gamma$ = 1-form *but in general not meromorphic*

Def: Generalized cycles $\quad \mathfrak{M}_1(\mathcal{S}) := \{\gamma \in \mathfrak{M}^1(\mathcal{S})^* \mid \Omega_\gamma \in \mathfrak{M}^1(\mathcal{S})\} \qquad \subset \mathfrak{M}^1(\mathcal{S})^*$

Map $\quad \hat{B} : \mathfrak{M}_1(\mathcal{S}) \to \mathfrak{M}^1(\mathcal{S})$

$$\gamma \mapsto \Omega_\gamma = < \gamma, B > = \int_\gamma B$$

Exact sequence $\quad 0 \to \mathrm{Ker}\, \hat{B} \to \mathfrak{M}_1(\mathcal{S}) \to \mathfrak{M}^1(\mathcal{S}) \to 0$

Def: Symplectic structure: $\quad \gamma_1 \cap \gamma_2 := \frac{1}{2\pi i} \left(\int_{\gamma_1} \Omega_{\gamma_2} - \int_{\gamma_2} \Omega_{\gamma_1} \right)$

Remark: if $\gamma_1, \gamma_2 \in H_1(\Sigma, \mathbb{C})$ $\qquad \gamma_1 \cap \gamma_2 =$ usual intersection

$\mathrm{Ker}\, \hat{B}$ is Lagrangian $\qquad \mathfrak{M}^1(\mathcal{S}) \sim \mathfrak{M}_1(\mathcal{S}) / \mathrm{Ker}\, \hat{B}$

Tangent space and cycles

Tangent space $T_{\mathcal{S}}\mathbb{S} = \mathfrak{M}^1(\mathcal{S}) \oplus \mathfrak{M}^1(\mathcal{S}) \overset{\text{sym}}{\otimes} \mathfrak{M}^1(\mathcal{S})$

Map $\hat{B} : \mathfrak{M}_1(\mathcal{S}) \to \mathfrak{M}^1(\mathcal{S})$

Def: from cycles to tangent vectors

$$\partial : \quad \mathfrak{M}_1(\mathcal{S}) \oplus \mathfrak{M}_1(\mathcal{S}) \overset{\text{sym}}{\otimes} \mathfrak{M}_1(\mathcal{S}) \to T_{\mathcal{S}}\mathbb{S}$$

$$\gamma \mapsto \quad \partial_\gamma \eta = \Omega_\gamma = \hat{B}(\gamma) = \int_\gamma B$$

$$\partial_\gamma B = \int_\gamma \omega_{0,3}$$

$$\frac{1}{2}(\gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1) \mapsto \quad \partial_{\gamma_1 \otimes \gamma_2} \eta = 0$$

$$\partial_{\gamma_1 \otimes \gamma_2} B = \frac{1}{2}\left(\Omega_{\gamma_1} \otimes \Omega_{\gamma_2} + \Omega_{\gamma_2} \otimes \Omega_{\gamma_1}\right)$$

Theorem: $$\partial_\gamma \omega_{g,n} = \int_\gamma \omega_{g,n+1}$$

$$\partial_{\gamma_1 \otimes \gamma_2} \omega_{g,n} = \int_{\gamma_1} \int_{\gamma_2} \left(\omega_{g-1,n+2} + \sum_{g_1+g_2=g,\ I_1 \sqcup I_2 = \{z_2,\ldots,z_n\}} \omega_{g_1,1+|I_1|}(\,.\,,I_1)\omega_{g_2,1+|I_2|}(\,.\,,I_2)\right)$$

Insertion operator

Hirota derivative
Insertion operator

Tangent space $T_{\mathcal{S}}\mathbb{S} = \quad \mathfrak{M}^1(\mathcal{S}) \oplus \mathfrak{M}^1(\mathcal{S}) \overset{\text{sym}}{\otimes} \mathfrak{M}^1(\mathcal{S})$ Map $\hat{B} : \mathfrak{M}_1(\mathcal{S}) \to \mathfrak{M}^1(\mathcal{S})$

$z \in \Sigma$ *choose a local coordinate* $\zeta(z)$

Def: of the cycle $\mathcal{B}_{z,1}$ $z \in \Sigma$ defined by $< \mathcal{B}_{z,1}, \omega > = \operatorname{Res}_{z' \to z} \frac{1}{\zeta(z) - \zeta(z')} \omega(z')$

$\Delta_z := \partial_{\mathcal{B}_{z,1}} \otimes d\zeta(z) \quad = \mathrm{ev}_z$ *(independent of a choice of local coordinate* $\zeta(z)$ *)*

1-form valued tangent vector

$\Delta \in H^0(\Sigma, K_\Sigma \otimes T_{\mathcal{S}}\mathbb{S})$

Theorem: $\Delta_z \omega_{g,n}(z_1, \ldots, z_n) = \omega_{g,n+1}(z_1, \ldots, z_n, z)$ $\Delta \omega_{g,n} = \omega_{g,n+1}$

(Trivial corollary of the previous)

Examples, Hirota derivative

Example $\mathcal{S}$

$$\Sigma = \mathbb{C} \qquad \pi(z) = z^2 \qquad \eta(z) = 2z^2\left(1 - \frac{1}{2}\sum_{k=1}^{\infty} t_k z^{2k}\right)dz \qquad B(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} dz_1 \otimes dz_2$$

$$\omega_{g,n}(z_1, \ldots, z_n) = (-1)^n \; 2^{2-2g-n} \sum_{d_1,\ldots,d_n} \left\langle e^{\frac{1}{2}\sum_k (2k+1)!!t_k \tau_{k+1}} \tau_{d_1} \ldots \tau_{d_n} \right\rangle_g \prod_{i=1}^{n} \frac{(2d_1+1)!! \; dz_i}{z_i^{2d_i+2}}$$

Remark: $t_k = \frac{1}{2\pi i}\int_{\mathcal{A}_{\infty,2k+3}} \eta \qquad \frac{\partial}{\partial t_k} = \partial_{\mathcal{B}_{\infty,2k+3}}$

Insertion operator: For large z do Taylor expansion *choose local coordinate* $\zeta(z) = z$

$$\frac{1}{z - z'} = \sum_{k=0}^{\infty} \frac{z'^k}{z^{k+1}} \qquad \Longrightarrow \qquad \Delta_z = \sum_{k=0}^{\infty} \frac{k dz}{z^{k+1}} \partial_{\mathcal{B}_{\infty,k}}$$

$$\Delta_z = \sum_{k=1}^{\infty} \frac{dz}{z^{2k+2}}(2k+1)\frac{\partial}{\partial t_{k-1}} + \textbf{even}$$

= usual form of Hirota operator for **KdV** or **KP**

Quantum Airy structures

[Kontsevich-Soibelman 2017]

Quantum Airy structure

Def: Partition function $Z(\hbar^{-1}\mathcal{S},\gamma) := e^{\sum_{(g,n)\neq(0,0)} \frac{\hbar^{2g-2+n}}{n!}\int_\gamma \dots \int_\gamma \omega_{g,n}(\mathcal{S})}$

Def: Tensors A,B,C,D $V = \mathscr{L} \subset \mathfrak{M}_1(\mathcal{S})$ Flat Lagrangian of $\mathfrak{M}_1 \to \mathbb{S}$ $V^* \sim \mathfrak{M}^1(\mathcal{S})$

$$C(\gamma,\omega,\omega') = \int_\gamma K(\omega,\omega') \qquad B(\gamma,\gamma',\omega) = \int_\gamma K(\hat{B}(\gamma'),\omega) = C(\gamma,\hat{B}(\gamma'),\omega) \qquad V \overset{\hat{B}}{\sim} V^*$$

$$A(\gamma_1,\gamma_2,\gamma_3) = \int_{\gamma_1}\int_{\gamma_2}\int_{\gamma_3} \omega_{0,3} = C(\gamma_1,\hat{B}(\gamma_2),\hat{B}(\gamma_3)) \qquad D(\gamma) = \int_\gamma \omega_{1,1} = C(\gamma,\omega_{0,2})$$

Theorem: $L = \hbar\Delta - \hbar\left(\omega_{1,1} + \hbar^2 K.\Delta\otimes\Delta\right)$ = quadratic differential operator

$$(L(z) - \Omega_\gamma(z)) \,.\, Z(\hbar^{-1}\mathcal{S},\gamma) = 0$$

$\gamma' \in V$ $L_{\gamma'} = \hbar\partial_{\gamma'} - \hbar\left(\int_{\gamma'}\omega_{1,1} + \hbar^2\int_{\gamma'} K.\Delta\otimes\Delta\right)$ = quadratic differential operator

$$\left(L_{\gamma'} - \int_{\gamma'}\Omega_\gamma\right) . \, Z(\hbar^{-1}\mathcal{S},\gamma) = 0$$

Quantum Airy structure

In a basis

Def: Partition function $Z(\hbar^{-1}\mathcal{S},\gamma) := e^{\sum_{(g,n)\neq(0,0)} \frac{\hbar^{2g-2+n}}{n!} \int_\gamma \cdots \int_\gamma \omega_{g,n}(\mathcal{S})}$

In a « basis » of cycles: $\gamma = \sum_k x_k \gamma_k$ $\quad Z(\hbar^{-1}\mathcal{S},\gamma) := e^{\sum_{(g,n)\neq(0,0)} \frac{\hbar^{2g-2+n}}{n!} \sum_{i_1,\ldots,i_n} x_{i_1}\ldots x_{i_n} F_{g,n}[i_2,\ldots,i_n]}$

where $F_{g,n}[i_2,\ldots,i_n] = \int_{\gamma_{i_1}} \cdots \int_{\gamma_{i_1}} \omega_{g,n} = \partial_{\gamma_{i_1}} \ldots \partial_{\gamma_{i_1}} F_g$ $\quad$ we « almost » have $\partial_{\gamma_i} \sim \frac{\partial}{\partial x_i}$

Def: Tensors A,B,C,D

$$A[i_1,i_2,i_3] := A(\gamma_{i_1},\gamma_{i_2},\gamma_{i_3}) = \int_{\gamma_{i_1}} \int_{\gamma_{i_2}} \int_{\gamma_{i_3}} \omega_{0,3} = F_{0,3}[i_1,i_2,i_3] \qquad D[k] := D(\gamma_k) = \int_{\gamma_k} \omega_{1,1} = F_{1,1}[k]$$

$$L_k = L_{\gamma_k} = \hbar \frac{\partial}{\partial x_k} - \hbar \left(D[k] + \hbar^2 \int_{\gamma'} K \,.\, \Delta \otimes \Delta \right)$$ = quadratic differential operator

Kontsevich Soibelman [2017]

$$L_k = \hbar \frac{\partial}{\partial x_k} - \hbar \left(D[k] + \frac{1}{2} \sum_{i,j} A[k,i,j] x_i x_j + 2B[k,i,j] x_i \hbar \frac{\partial}{\partial x_j} + C[k,i,j] \hbar \frac{\partial}{\partial x_i} \hbar \frac{\partial}{\partial x_j} \right)$$

Higher ramification order

Spectral curve $\quad \text{order } \pi(a) = r_a \geq 2 \quad \pi(z) = \pi(a) + z^{r_a} \quad \rho = e^{2\pi i/r_a}$

Def: Topological Recursion

$\omega_{0,1}(\mathcal{S}) = \eta \qquad \omega_{0,2}(\mathcal{S}) = B \qquad K_a^{(k)}(z_1, \zeta_1, \ldots, \zeta_k) = \frac{\int_a^z \omega_{0,2}(z_1, .)}{\prod_{j=2}^k \omega_{0,1}(\zeta_1) - \omega_{0,1}(\zeta_j)}$

$2g - 2 + n > 0 \quad , n \geq 1$

$$\omega_{g,n}(\mathcal{S}; z_1, \ldots, z_n) = \sum_{a \in R} \sum_{k=2}^{r_a} \sum_{\sigma \subset_{k-1} \{1,\ldots,r_a-1\}} \operatorname*{Res}_{z \to a} K_a^{(k)}(z_1, z, \rho^{\sigma_1} z \ldots, \rho^{\sigma_{k-1}} z) \Big[\sum_{\mu \vdash \sigma \cup \{0\}} \sum_{I_1 \sqcup I_2 \ldots \sqcup I_{\ell(\mu)} = \{z_2, \ldots, z_n\}} \sum_{g_1 + \ldots + g_\ell = g-k+\ell}^{no\ (0,1)} \prod_{i=1}^{\ell} \omega_{g_i, |\mu_i| + |I_i|}(\mu_i, I_i) \Big]$$

Operator notation: $\quad K^{(k)} : \mathfrak{M}^1(\mathcal{S})^{\otimes k} \to \mathfrak{M}^1(\mathcal{S})$

$$\omega_{g,n} = \sum_{k=2}^{r} K^{(k)} \Big[\sum_{\sigma \subset_{k-1} \{1,\ldots,r_a-1\}} \sum_{\mu \vdash \sigma \cup \{0\}} \sum_{I_1 \sqcup I_2 \ldots \sqcup I_{\ell(\mu)} = \{z_2, \ldots, z_n\}} \sum_{g_1 + \ldots + g_\ell = g-k+\ell}^{no\ (0,1)} \prod_{i=1}^{\ell} \omega_{g_i, |\mu_i| + |I_i|}(\mu_i, I_i) \Big]$$

Higher Quantum Airy structure

Spectral curve order $\pi(a) = r_a \geq 2$ $\pi(z) = \pi(a) + z^{r_a}$ $\rho = e^{2\pi i/r_a}$

Theorem:

$$L = \hbar\Delta - \hbar\omega_{0,1} - \hbar\sum_{k=2}^{r} \hbar^k \; K^{(k)}(\Delta^k)$$

r-th order differential operator

$$(L(z) - \Omega_\gamma(z)) \, . \, Z(\hbar^{-1}\mathcal{S}, \gamma) = 0$$

annihilates the partition function

Virasoro W-algebra

Virasoro and W constraints

Def: $\mathfrak{W}$ algebra generators:

If Σ is a degree r cover of Σ_0 $\quad x \in \Sigma_0 \quad \#\pi^{-1}(x) = r$

$$\mathfrak{W}_k(x) := \pi_* \sum_{I \subset_k \pi^{-1}(x)} \prod_{z \in I} \Delta_z \qquad \mathfrak{W}_0(x) := 1 \qquad \mathfrak{W}_k(x) = 0 \quad \text{if} \quad k > r$$

$$\mathfrak{W}_k \in H^0(\Sigma_0, K_{\Sigma_0}^k \boxtimes T_{\mathcal{S}} \mathbb{S}^{\otimes k}) \qquad \mathfrak{W}(x,y) = \sum_{k=0}^{r} (-1)^k y^{r-k} \mathfrak{W}_k(x) \qquad y \in T_x^* \Sigma_0$$

k-th order form valued k-th order differential operator

Examples: $\mathfrak{W}_1(x) = \sum_{z \in \pi^{-1}(x)} \pi_* \Delta_z \qquad \mathfrak{W}_2(x) = \sum_{z \neq z' \in \pi^{-1}(x)} \pi_* \Delta_z \Delta_{z'}$ ... etc

Examples:

$$\mathfrak{W}_1(x) . \omega_{g,n}(z_1, \ldots, z_n) = \sum_{z \in \pi^{-1}(x)} \omega_{g,n+1}(z, z_1, \ldots, z_n)$$

$$\mathfrak{W}_2(x) . \omega_{g,n}(z_1, \ldots, z_n) = \sum_{z \neq z' \in \pi^{-1}(x)} \omega_{g,n+2}(z, z', z_1, \ldots, z_n)$$

Theorem: **Loop equations = Virasoro W-algebra constraints**

$$P_k(x; z_1, \ldots, z_n) = \mathfrak{W}_k(x) . \Delta_{z_1} \ldots \Delta_{z_n} e^{\sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\mathcal{S})}$$

has no poles at branchpoints

= analytic at $x = \pi(a)$

Virasoro and W constraints

$$\mathfrak{W}_k(x) := \pi_* \sum_{I \subset_k \pi^{-1}(x)} \prod_{z \in I} \Delta_z \qquad \mathfrak{W}(x,y) = \sum_{k=0}^{r} (-1)^k y^{r-k} \mathfrak{W}_k(x) \qquad y \in T_x^* \Sigma_0$$

$$P'(z) := \prod_{z' \in \pi^{-1}(\pi(z)) \setminus z} (\eta(z) - {}^*\eta(z')) \qquad P' \in H^0(\Sigma, K_\Sigma^{r-1})$$

Theorem: Loop equations , Virasoro, W constraints

$$P_k(x; z_1, \ldots, z_n) = \mathfrak{W}_k(x) \,.\, \Delta_{z_1} \ldots \Delta_{z_n} e^{\sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\mathcal{S})}$$ has no poles at branchpoints

= analytic at $x = \pi(a)$

$$P(z; z_1, \ldots, z_n) = \frac{1}{P'(z)} \mathfrak{W}(x,y) \,.\, \Delta_{z_1} \ldots \Delta_{z_n} e^{\sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\mathcal{S})} \Big|_{x=\pi(z), y=\eta(z)}$$

is a 1-form of z analytic at ramification points

Virasoro and W constraints

$$\mathfrak{W}_k(x) := \pi_* \sum_{I \subset_k \pi^{-1}(x)} \prod_{z \in I} \Delta_z \qquad \mathfrak{W}(x,y) = \sum_{k=0}^{r} (-1)^k y^{r-k} \mathfrak{W}_k(x) \qquad y \in T_x^* \Sigma_0$$

$$P'(z) := \prod_{z' \in \pi^{-1}(\pi(z)) \setminus z} (\eta(z) - {}^*\eta(z')) \qquad P' \in H^0(\Sigma, K_\Sigma^{r-1})$$

Theorem: Loop equations , Virasoro, W constraints

$$P_k(x; z_1, \dots, z_n) = \mathfrak{W}_k(x) \,.\, \Delta_{z_1} \dots \Delta_{z_n} e^{\sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\mathcal{S})}$$ has no poles at branchpoints

= analytic at $x = \pi(a)$

$$P(z; z_1, \dots, z_n) = \frac{1}{P'(z)} \mathfrak{W}(x,y) \,.\, \Delta_{z_1} \dots \Delta_{z_n} e^{\sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\mathcal{S})} \Big|_{x=\pi(z), y=\eta(z)}$$

is a 1-form of z analytic at ramification points

Theorem: $\forall \gamma \in H_1(\Sigma, \mathbb{C})$ $Z(\hbar^{-1}\mathcal{S}, \gamma)$ = solution of loop equations Virasoro-W constraints

$$\sum_{\gamma \in H_1(\Sigma, \mathbb{C})} c_\gamma \, Z(\hbar^{-1}\mathcal{S}, \gamma)$$ general solution of loop equations Virasoro-W constraints

Theorem: $\mathcal{T}(\hbar^{-1}\mathcal{S}) = \sum_{\gamma \in H_1(\Sigma, \mathbb{Z})} Z(\hbar^{-1}\mathcal{S}, \gamma)$ = modular invariant solution of loop equations

Wave-function

Sato and Hirota relation

Let D = divisor on Σ $(\deg D = 0)$

3rd kind cycle = open chain γ_D with boundary $\partial\gamma_D = D$

« would-be wave function » : $\dfrac{Z(\hbar^{-1}\mathcal{S},\gamma_D)}{Z(\hbar^{-1}\mathcal{S})} = e^{\sum_{g,n}\frac{\hbar^{2g-2+n}}{n!}\int_{\gamma_D}\cdots\int_{\gamma_D}\omega_{g,n}}$

Problem: depends on γ_D rather than D

Def: Wave function, Baker-Akhiezer function

$$\Psi(\hbar^{-1}\mathcal{S};D) := \sum_{\gamma\in H_1(\Sigma,\mathbb{Z})} e^{\sum_{g=1}^{\infty}\hbar^{2g-2}F_g(\mathcal{S}+\gamma_D+\gamma)} = \sum_{\gamma,\ \partial\gamma=D} e^{\sum_{g=1}^{\infty}\hbar^{2g-2}F_g(\mathcal{S}+\gamma)}$$

(defined as a trans-series in powers of $\hbar$ and powers of $e^{\hbar^{-1}}$)

Def: **Hirota equation**

We say that Hirota equation is satisfied iff

$$\mathcal{T}(\hbar^{-1}\mathcal{S})\ \Psi(\hbar^{-1}\mathcal{S};[p_1]-[p_2]+[p_3]-[p_4]) = \Psi(\hbar^{-1}\mathcal{S};[p_1]-[p_2])\ \Psi(\hbar^{-1}\mathcal{S};[p_3]-[p_4]) - \Psi(\hbar^{-1}\mathcal{S};[p_1]-[p_4])\ \Psi(\hbar^{-1}\mathcal{S};[p_3]-[p_2])$$

Theta functions

$\Lambda = \mathscr{L} \cap H_1(\Sigma, \mathbb{Z})$ = integer lattice of cycles , transverse to $\mathrm{Ker}\,\hat{B}$

Def: **Theta function**

$$\Theta(\omega) := \sum_{n\in\Lambda} e^{<n,\omega>} e^{\frac{1}{2}<n,\hat{B}(n)>}$$

and its derivatives

$$\Theta^{(k)}(\omega) := \sum_{n\in\Lambda} e^{<n,\omega>} e^{\frac{1}{2}<n,\hat{B}(n)>} \overbrace{n \otimes n \otimes \ldots \otimes n}^{k} \in \Lambda^{\otimes k}$$

Def: **Tau function**

$$\mathscr{T}(\hbar^{-1}\mathcal{S}) = \sum_{n\in\Lambda} Z(\hbar^{-1}\mathcal{S} + n)$$

$$= e^{\sum_g \hbar^{2g-2} F_g(\mathcal{S})} \Big(\Theta(\hbar^{-1}\eta) + \hbar(< \Theta'(\hbar^{-1}\eta), \omega_{1,1} > + \frac{1}{6} < \Theta'''(\hbar^{-1}\eta), \omega_{0,3} >) + O(\hbar^2 \Big)$$

(defined as a trans-series in powers of $\hbar$ and powers of $e^{\hbar^{-1}}$)

Def: Wave function, Baker-Akhiezer function

$$\Psi(\hbar^{-1}\mathcal{S}; D) = \frac{\mathscr{T}(\hbar^{-1}\mathcal{S} + \gamma_D)}{\mathscr{T}(\hbar^{-1}\mathcal{S})}$$

Back

CFT

CFT

Def: **CFT notations**

$$\left\langle \mathscr{V}(\hbar^{-1}\mathcal{S}) \right\rangle = Z(\hbar^{-1}\mathcal{S}) = e^{\sum_g \hbar^{2g-2} F_g(\mathcal{S})}$$

Sugawara current $J(z) = \Delta_z$

$$\left\langle J(z)\ \mathscr{V}(\hbar^{-1}\mathcal{S}) \right\rangle = e^{\sum_g \hbar^{2g-2} F_g(\mathcal{S})} \sum_{g=0}^{\infty} \hbar^{2g-1} \omega_{g,1}(z)$$

W-algebra

$\mathfrak{W}_k(x) =$ k th Casimir of J

Vertex operator $\mathscr{V}_\alpha(z)$ of charge α at point z

Divisor $D = \alpha_1[z_1] + \ldots + \alpha_n[z_n]$

$$\left\langle \mathscr{V}(\hbar^{-1}\mathcal{S}) \mathscr{V}_{\alpha_1}(z_1) \ldots \mathscr{V}_{\alpha_n}(z_n) \right\rangle = \left\langle \mathscr{V}(\hbar^{-1}\mathcal{S} + \gamma_D) \right\rangle$$

Theorem: **Satisfy OPE and Ward Identities of a CFT**

$$J(z_1)J(z_2) \sim \frac{dz_1 dz_2}{(z_1 - z_2)^2} \qquad \left\langle J(z_1)J(z_2)\ \mathscr{V}(\hbar^{-1}\mathcal{S}) \right\rangle \sim \frac{dz_1 dz_2}{(z_1 - z_2)^2} \left\langle \mathscr{V}(\hbar^{-1}\mathcal{S}) \right\rangle$$

$$= e^{\sum_g \hbar^{2g-2} F_g(\mathcal{S})} \left(\sum_{g=0}^{\infty} \hbar^{2g} \omega_{g,2}(z_1, z_2) + \sum_{g,h} \hbar^{2g+2h-2} \omega_{g,1}(z_1)\omega_{h,1}(z_2) \right) \sim \omega_{0,2}(z_1, z_2) e^{\sum_g \hbar^{2g-2} F_g(\mathcal{S})} + \ldots$$

Conclusion

- **Topological recursion:** Associates sequence of forms to spectral curves

  Plenty of applications: enumerative geometry, Gromov-Witten, Random matrices, maps,…

- **Deformations:** Cycles -> tangent vectors

  Defines symplectic structures for deformations (hyperKaehler, Goldman), Intrinsic

  Avoids to have « infinite number of times »

- **Using cycles:** Intrinsic way of writing

  Virasoro, W-algebra, loop equations

  Quantum Airy structures

  Hirota, Sato relations

- **CFT:** T.R. defines a (perturbative) CFT for any spectral curve

Examples

Example KdV

$$\mathcal{S} \quad \begin{cases} \Sigma = \mathbb{C} \\ \pi(z) = z^2 \qquad \text{ramification point} \quad a = 0 \qquad \text{Involution} \quad \sigma_a(z) = -z \\ \eta(z) = 2z^2 \left(1 - \frac{1}{2} \sum_{k=1}^{\infty} t_k z^{2k} \right) dz \\ B(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} dz_1 \otimes dz_2 \end{cases}$$

$\omega_{g,n}$ —> Witten-Kontsevich Intersection numbers

$$\omega_{g,n}(z_1, \ldots, z_n) = (-1)^n \, 2^{2-2g-n} \sum_{d_1,\ldots,d_n} \left\langle e^{\frac{1}{2}\sum_k (2k+1)!!t_k\tau_{k+1}} \tau_{d_1} \ldots \tau_{d_n} \right\rangle_g \prod_{i=1}^{n} \frac{(2d_i+1)!! \, dz_i}{z_i^{2d_i+2}}$$

Witten-Kontsevich « theorem » : ψ **= KdV**