# The resurgent structure of quantum knot invariants (and topological strings)

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# The hidden structure of perturbation theory

Perturbation theory in a small parameter remains one of the most fruitful approaches to quantum theories.

Thanks to the theory of resurgence, we have learned that a perturbative series encodes much more information than originally thought. By an appropriate decoding of such a series, one can find e.g. additional sectors of the path integral.

This decoding involves beautiful and challenging mathematics, in which one passes from formal power series to analytic complex functions and their underlying geometry.

As I will show in this talk, the resurgent structure of perturbative series involves in some cases a hidden **integrality structure** which can be physically interpreted in terms of counting BPS states in a "dual" theory.

Two very interesting examples of such a situation are complex Chern-Simons theory and topological string theory.

This is based on 2007.10190 and in progress with Stavros Garoufalidis and Jie Gu (for knots), as well as work in progress with Jie Gu (for topological strings).

# The Borel triangle

The Borel method is a systematic (and traditional) way of making sense of factorially divergent formal power series

$$\varphi(z) = \sum_{n\geq 0} a_n z^n$$

Borel transform

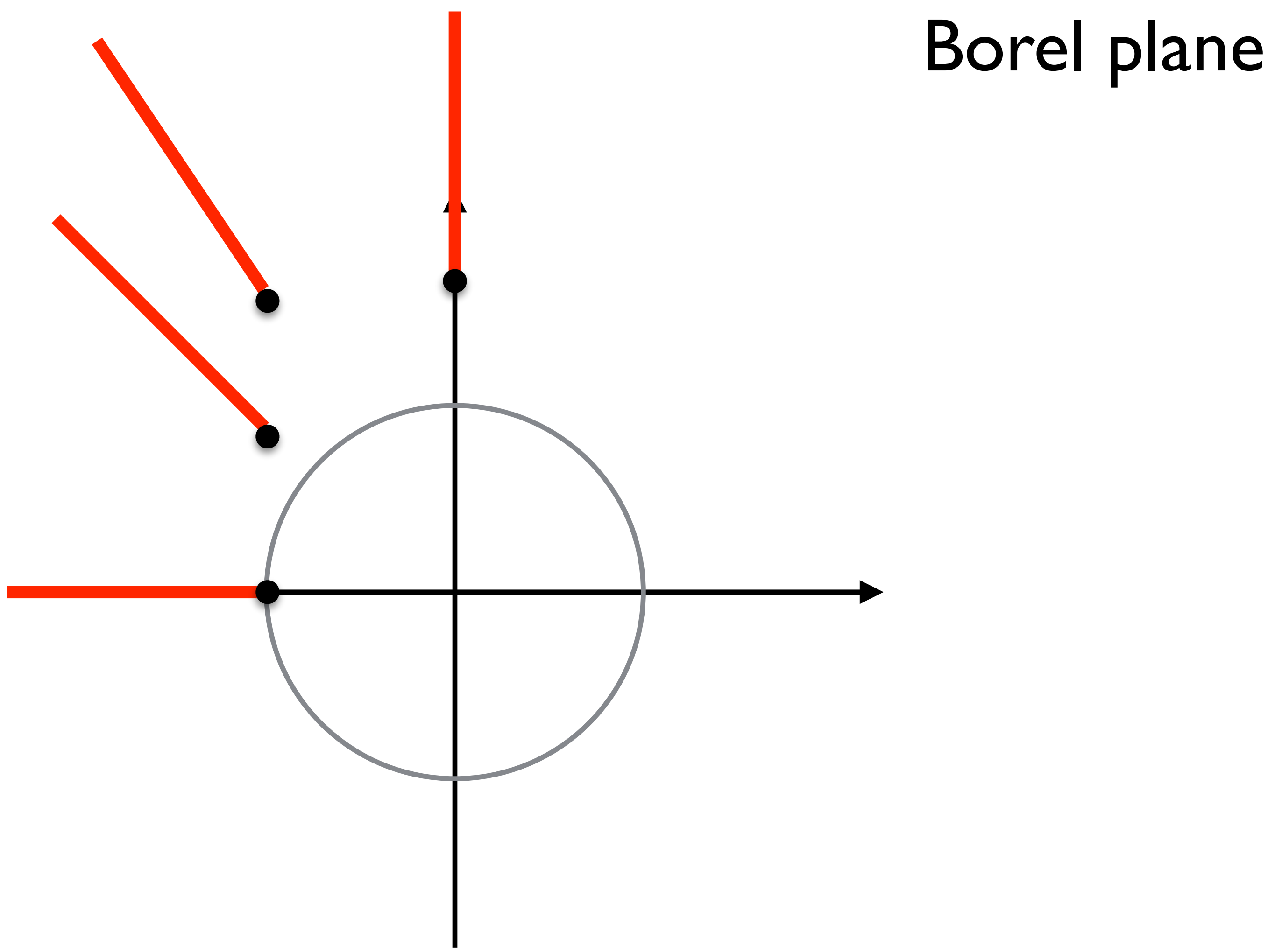
$$\widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!}\zeta^n$$

Laplace transform

$$s(\varphi)(z) = \int_0^\infty \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)\mathrm{d}\zeta$$

Borel resummation

The Borel transform $\widehat{\varphi}(\zeta)$ is analytic at the origin. Very often it can be analytically continued to the complex plane, displaying a set of **singularities** (poles, branch cuts)



The expansion of the Borel transform around each singularity leads to new formal power series. For so-called simple resurgent functions, we have only log singularities:

$$\widehat{\varphi}(\zeta) = -\mathsf{S}_\omega\, \widehat{\varphi}_\omega(\zeta - \zeta_\omega) \frac{\log(\zeta - \zeta_\omega)}{2\pi \mathrm{i}} + \cdots$$

Stokes constant

$$\varphi_\omega(z) = \sum_{n \ge 0} a_{n,\omega} z^n$$

These new perturbative series are typically associated to new sectors of the theory (e.g. expansions around different saddle points)

We can repeat the same analysis for all these new functions. We conclude that, starting from a perturbative series, we generate **a set of formal power series**, corresponding to the different sectors of the path integral, and **a matrix of Stokes constants**

$\varphi_\omega(z)$     $\mathsf{S}_{\omega\omega'}$     $\omega$ labels the sectors

Extracting all this information in a given physical theory is not easy, to put it mildly.

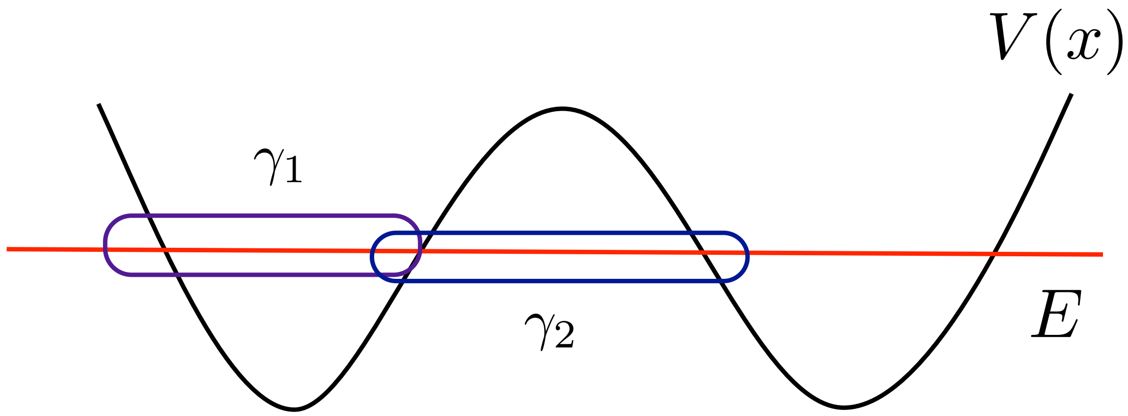
Stokes constants are often complicated numbers. However, in some cases they turn out to be **integers**. In simple examples this has a geometric interpretation in terms of intersection of integration paths (cf. Kontsevich's talk)

# Exact WKB and BPS states

In the exact WKB method, the formal power series are (exponentiated) quantum periods, or Voros symbols

"WKB curve"

$$\Sigma(x,p) = H(x,p) - E = 0$$




quantum periods

$$\Pi_\gamma(\hbar) = \oint_\gamma p(x,\hbar)\mathrm{d}x = \sum_{n\geq 0} \Pi_\gamma^{(n)} \hbar^{2n}$$

Voros symbols

$$\varphi_\gamma(\hbar) = \mathrm{e}^{\Pi_\gamma(\hbar)}$$

The Stokes constants for this problem are integers that can be computed with the exact WKB method [Voros, Ecalle, Delabaere-Dillinger-Pham…] and they depend on the moduli of the curve.

Suppose that we can regard the WKB curve $\Sigma$ as a **Seiberg-Witten curve** for a supersymmetric theory. One consequence of the work of Gaiotto-Moore-Neitzke (GMN) is that Stokes constants compute BPS multiplicities:

$$\mathsf{S}_{\gamma\gamma'} = \langle \gamma, \gamma' \rangle \Omega(\gamma')$$

# Complex Chern-Simons theory

A rich source of perturbative series (and beautiful mathematics!) is complex CS theory on a 3-manifold *M*.

When *M* is the complement of a hyperbolic knot *K*, it has been argued that the partition function of the theory can be reduced to a finite-dimensional **"state integral"** [Kashaev, Hikami, Dimofte et al., Andersen-Kashaev]

$$Z_K(\tau) = \int \mathrm{e}^{-W(\mathbf{x};\tau)/\tau} \mathrm{d}\,\mathbf{x} \qquad \tau \in \mathbb{C}\backslash(-\infty, 0]$$

The building block of the integrand is Faddeev's quantum dilogarithm. We will focus on *SL(2,C)*

Saddle-points correspond to flat complex connections $\sigma$ on *M*. The expansions around these connections have the form

$$\exp\left(\frac{V(\sigma)+\mathrm{i}\mathcal{C}(\sigma)}{2\pi\tau}\right)\varphi_\sigma(\tau)$$

$$\varphi_\sigma(\tau)=\sum_{n\geq 0}a_n^\sigma\tau^n$$

factorially divergent series!

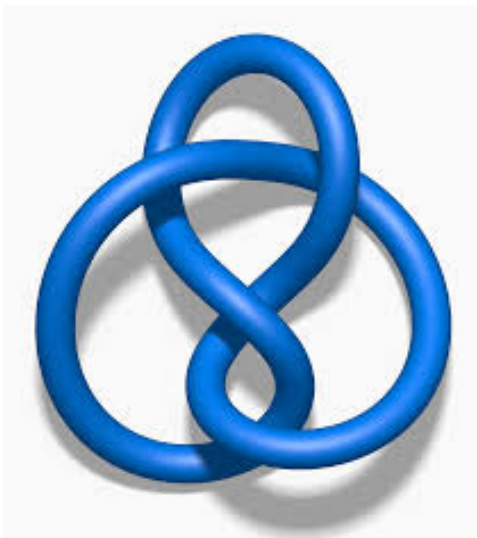
$$a_n^\sigma\sim n!$$

Among these connections there is always the "geometric connection" *g* (corresponding to the geodesically complete hyperbolic metric on *M*), and its conjugate *c*, with

$$V(g,c)=\pm V$$

hyperbolic volume

# These saddle points lead to “classical” singularities in the Borel plane.



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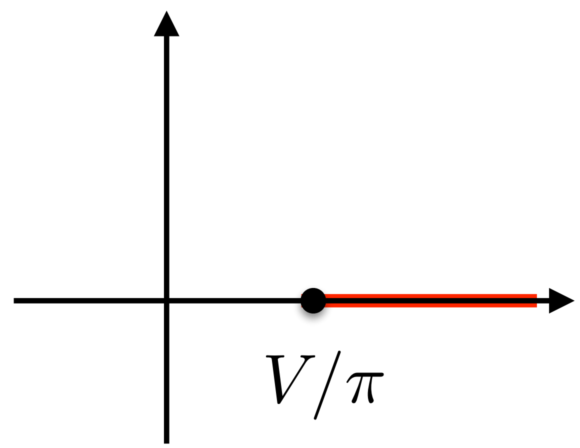
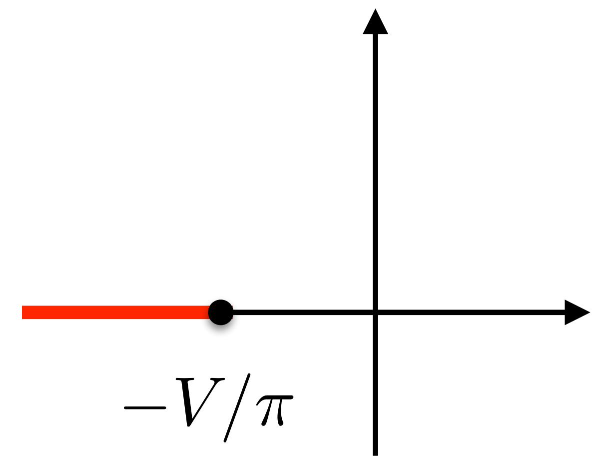



$\varphi_g(\tau)$




$\varphi_c(\tau)$

$$\varphi_g\left(\frac{\tau}{2\pi \mathrm{i}}\right) = \frac{1}{3^{1/4}}\left(1 + \frac{11\tau}{72\sqrt{-3}} + \frac{697\tau^2}{2\,(72\sqrt{-3})^2} + \frac{724351\tau^3}{30\,(72\sqrt{-3})^3} + \cdots\right)$$

=all-orders
Kashaev invariant

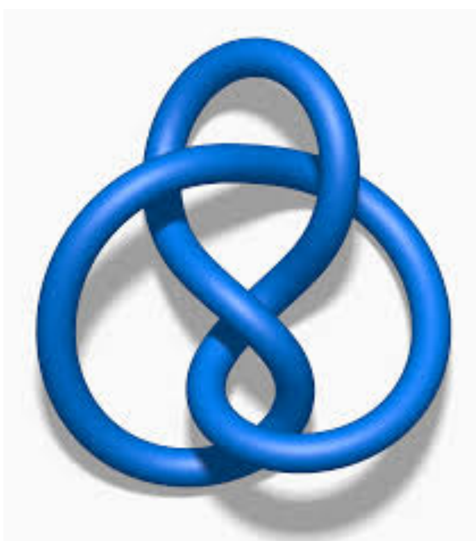
$$\varphi_c(\tau) = \varphi_g(-\tau)$$

These "classical" singularities and their Stokes constants (a finite number) were analyzed in e.g. [Gukov-M.M.-Putrov, Gang-Hatsuda, Garoufalidis-Zagier]

It turns however that, due to multivaluedness of the CS action and of the state integral potential, there are **infinite towers of additional singularities** [Garoufalidis, Witten, Gukov-M.M.-Putrov,...], corresponding to the "actions"

$$\frac{V(\sigma) + \mathrm{i}\mathcal{C}(\sigma)}{2\pi} + 2\pi \mathrm{i} n \qquad n \in \mathbb{Z}$$

The actual picture is rather



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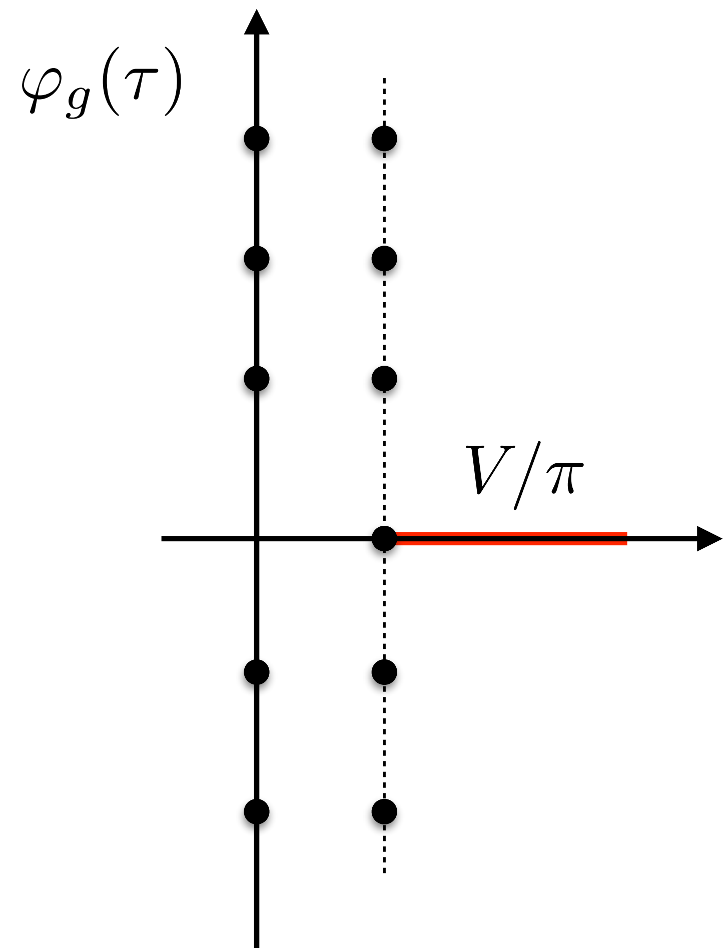
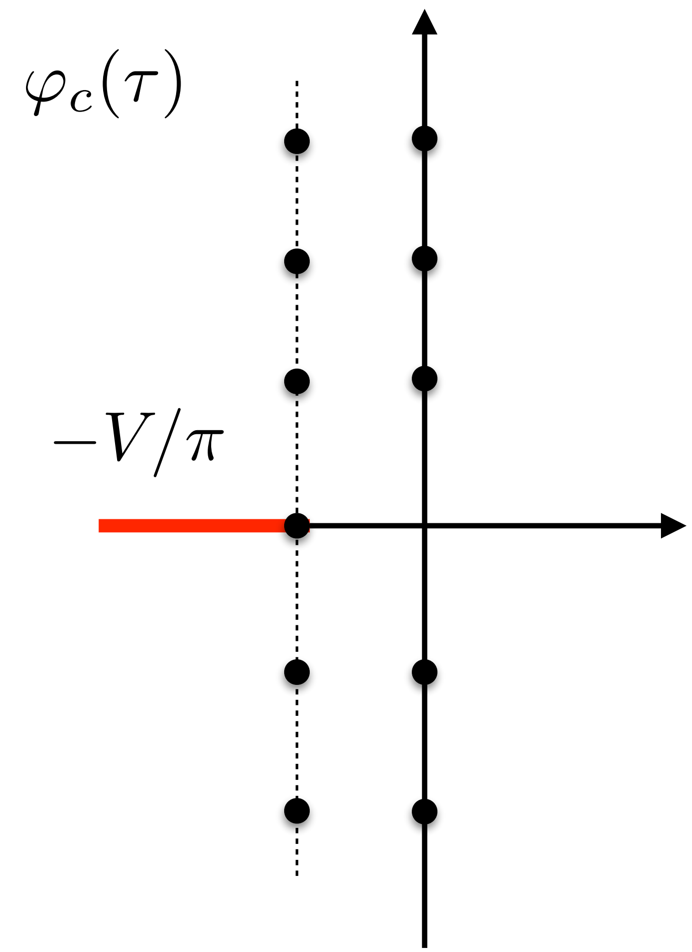



Similar infinite towers of singularities appear in other contexts, like topological string theory [Pasquetti-Schiappa, …]

These towers correspond to tiny non-perturbative corrections of order $\tilde{q}^n$

$$\tilde{q} = \exp\left(-\frac{2\pi \mathrm{i}}{\tau}\right)$$

We have, in principle, infinitely many Stokes constants, labeled by a pair of flat connections and a pair of integers

$$\omega = (\sigma, n) \qquad\qquad \mathsf{S}_{\omega\omega'} = \mathsf{S}_{\sigma,\sigma';n,n'}$$

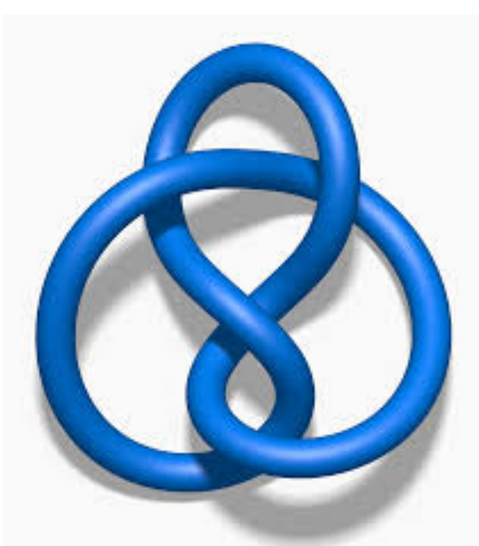
One might think that the towers of singularities are somewhat "trivial" since they come from multivaluedness. Indeed, the different formal power series have a simple multiplicative structure

$$\varphi_{\sigma,n}(\tau) = \varphi_{\sigma}(\tau)\tilde{q}^n$$

The Stokes constants depend then on a single integer *n*

$$\mathsf{S}_{\sigma,\sigma';n} = \mathsf{S}_{\sigma,\sigma';0,n}$$

and explicit computations show that they provide
**a highly non-trivial collection of integer invariants of the knot!**



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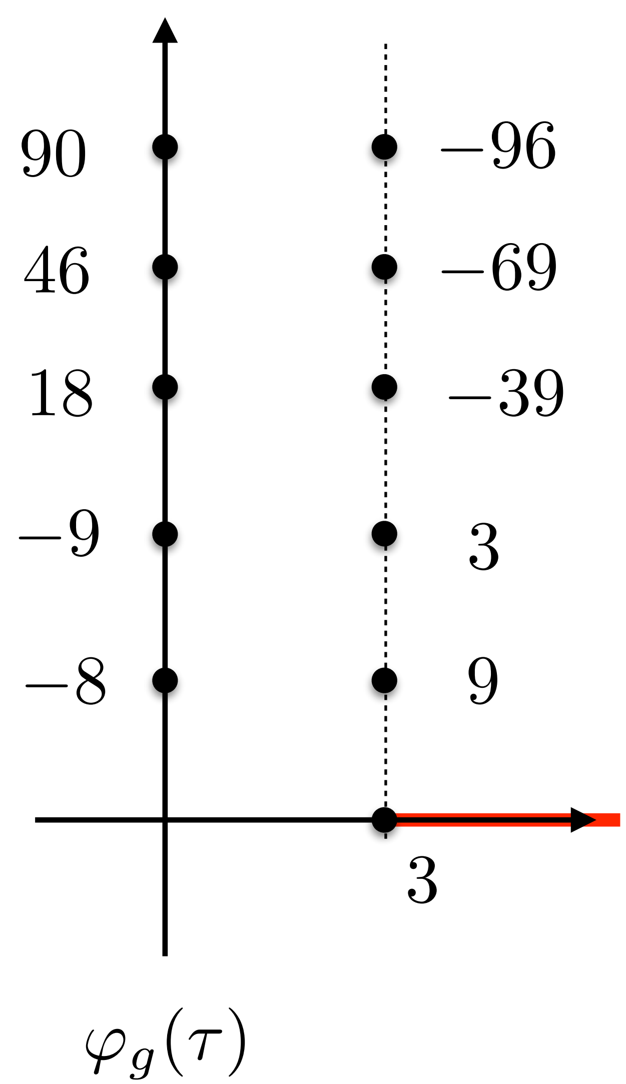
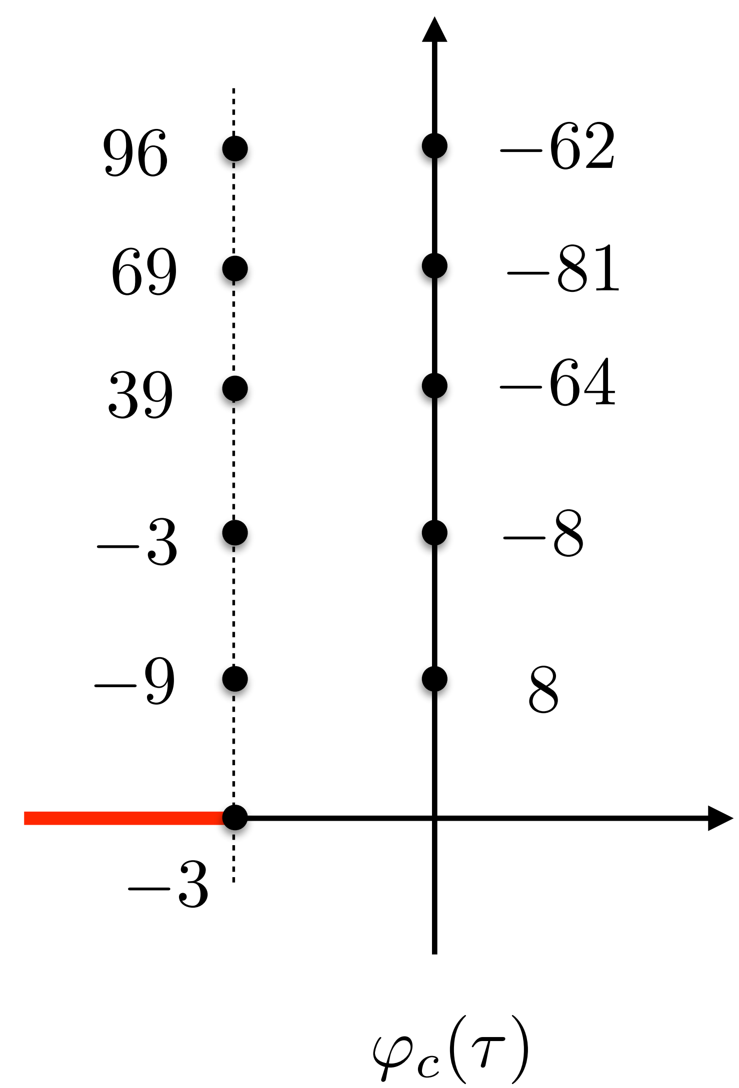
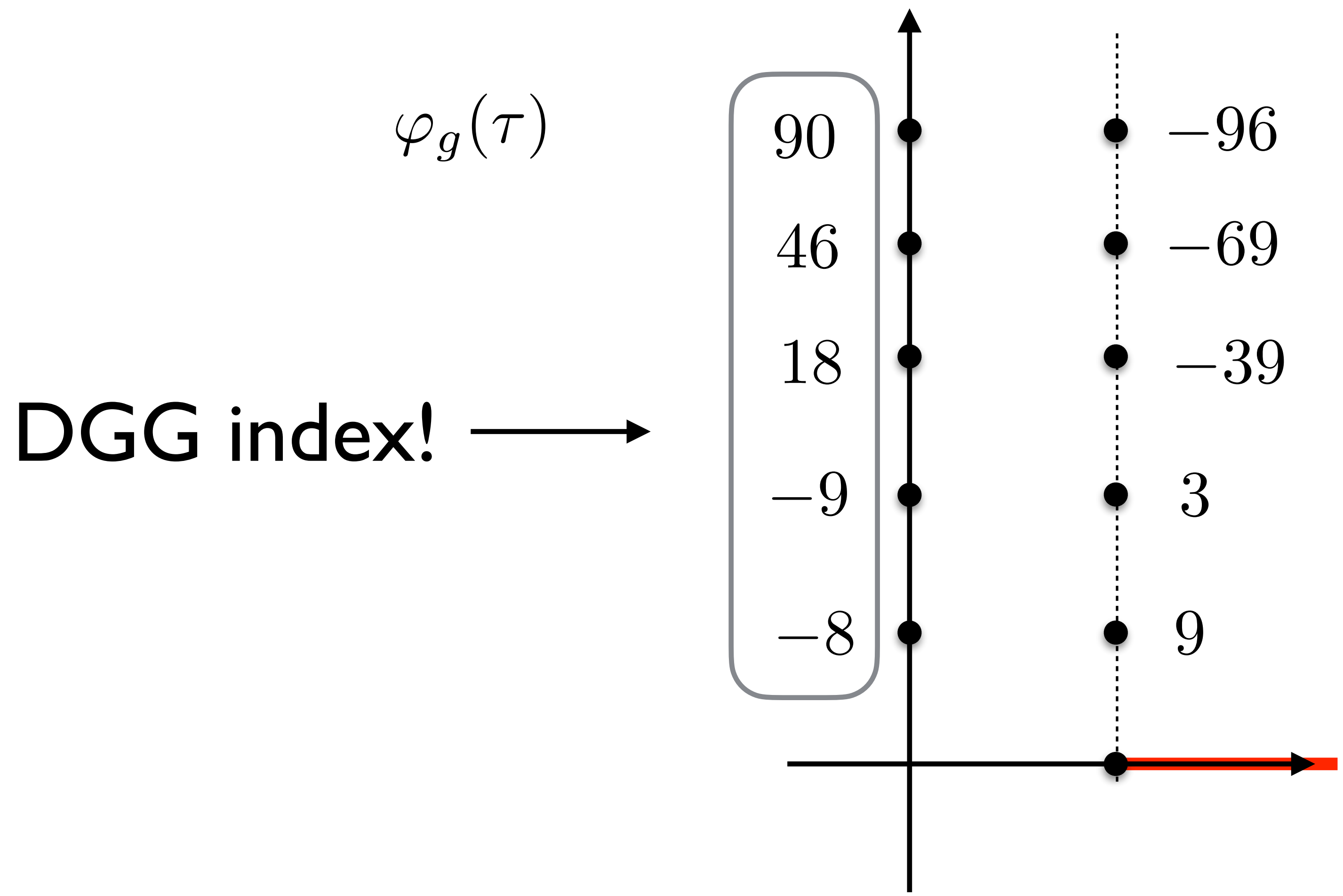


# Can we find a systematic description of these integers?




$$\mathrm{Tr}_{\mathcal{H}_{m=0}}(-1)^F q^{R/2+j_3} = 1 - 8q - 9q^2 + 18q^3 + 46q^4 + \cdots$$

This counts BPS states in a 3d SCFT "dual" to the hyperbolic knot [Dimofte-Gaiotto-Gukov]

A more precise description involves the *q*-series appearing in the "block decomposition" of the state integral [Garoufalidis-Kashaev, Garoufalidis-Zagier] and generalizations thereof.

Consider the linear *q*-difference equation:

$$y_{m+1}(q) + y_{m-1}(q) - (2 - q^m) y_m(q) = 0$$

Then, the "Stokes *q*-series"

$$\mathcal{S}^{\pm}_{\sigma\sigma'}(q) = \sum_{n \geq 1} \mathsf{S}_{\sigma,\sigma';n} q^{\pm n}$$

are given by explicit bilinear expressions in fundamental solutions of the above equation

# Explicit formulae

$$g_\lambda(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)+n\lambda}}{(q)_n^2}$$

$$G_\lambda(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)+n\lambda}}{(q)_n^2} \left( 2\lambda + E_1(q) + 2\sum_{j=1}^{n} \frac{1+q^j}{1-q^j} \right)$$

$$g_+(q) = g_0(q) \qquad g_-(q) = g_1(q) + 2g_{-1}(q)$$

$$G_+(q) = G_0(q) \qquad G_-(q) = -G_1(q) - 2G_{-1}(q)$$

$$\mathcal{S}^+(q) = \begin{pmatrix} g_+G_+ - 1 & 1 - g_-G_+ \\ 1 - g_+G_- & g_-G_- - 1 \end{pmatrix}$$

We have obtained similar explicit results for the $5_2$ knot.

# General conjectures

We conjecture that, for any hyperbolic knot:

1. $\mathcal{S}^{+}_{gg}(q) =$ DGG index $-1$

2. All the Stokes *q*-series are bilinear expressions in solutions to a linear difference equation, which is in turn determined by the "block decomposition" of the state integral.

Related results have been recently announced by Kontsevich

"Stokes q-series" were previously calculated for SU(2) CS on some Seifert spaces in [Costin-Garoufalidis, Gukov-M.M.-Putrov]

# Resurgent structures in topological strings

It turns out that similar structures appear in topological string theory on toric Calabi-Yau (CY) manifolds.

Given a toric CY $X$, we consider the partition function of the topological string in the **conifold** frame

$$Z(\lambda, \tau) = \exp \left( \sum_{g \geq 0} F_g(\lambda) \tau^{2g-2} \right)$$

$\tau$ =string coupling constant

We now define an **infinite family of formal power series** (times an exponential)

$$Z_N(\tau) = Z(\lambda = N\tau, \tau) \qquad N = 1, 2, \cdots$$

Motivation: due to the TS/ST correspondence [Grassi-Hatsuda-M.M.], these series are, conjecturally, asymptotic expansions of (fermionic) spectral traces of trace-class operators on the real line

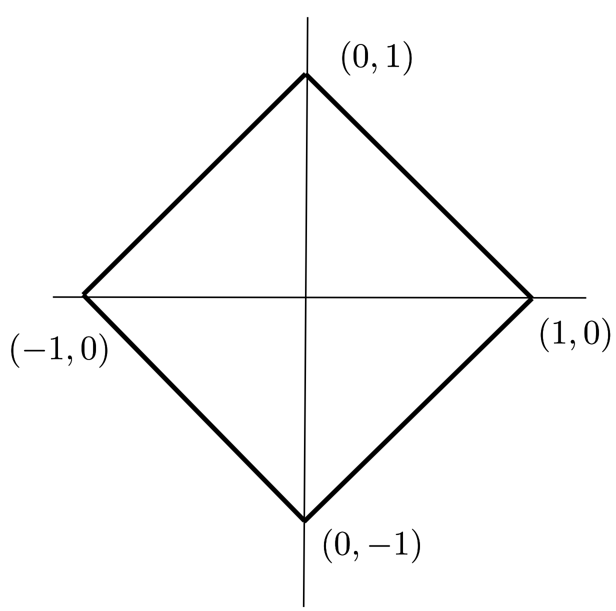
$$\mathrm{Tr}_{\Lambda^N(L^2(\mathbb{R}))} \rho_X^{\otimes N} \sim Z_N(\tau)$$

These operators are obtained by "quantizing" the mirror curve to *X*

What is the resurgent structure of these series?

It turns out that it is very similar in many cases to the one found in complex CS theory.

A rich example which can be worked out in detail: local $\mathbb{F}_0$




$$\rho_{\mathbb{F}_0}^{-1} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}$$

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$$

$$\tau = -\frac{1}{\hbar}$$

$$\mathrm{Tr}\,\rho_{\mathbb{F}_0} \sim \exp\left(\frac{4G}{2\pi\tau}\right)\underbrace{\left(1+\frac{\pi\tau}{24}+\cdots\right)}_{\varphi_g(\tau)}$$

"volume" *V* of the CY at the conifold point

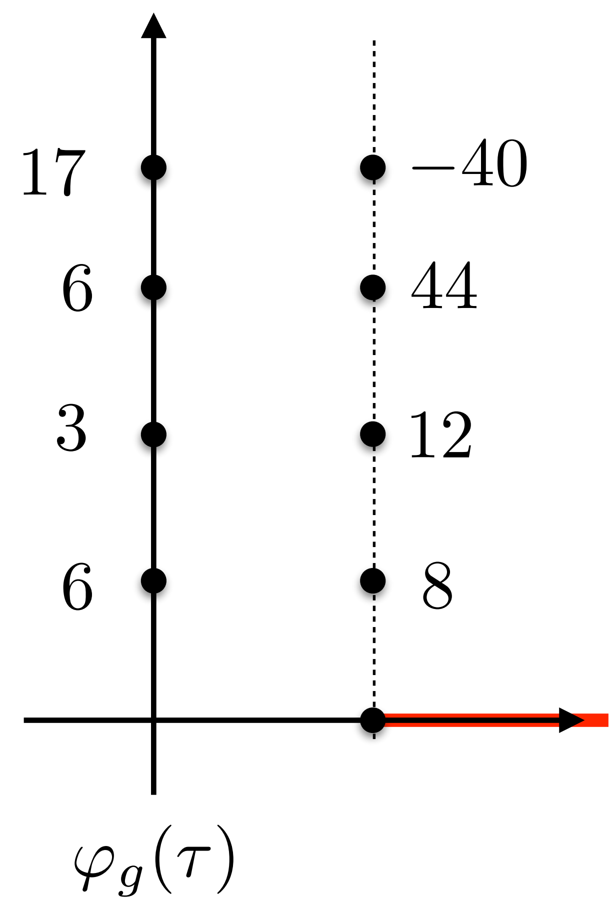
This is the **volume conjecture for toric CY manifolds**:

$$\mathrm{Tr}_{\Lambda^N(L^2(\mathbb{R}))}\rho_X^{\otimes N} \sim \exp\left(-N\hbar V\right)$$

It turns out that, as in the figure eight knot, there is another formal power series involved in the resurgent structure for *N=1*

$$\varphi_c(\tau) = \varphi_g(-\tau)$$

In this case, the additional towers of additional singularities are located at $\pi n \mathrm{i}$




We still don't know the physical or mathematical interpretation of these integers, which are certainly part of the topological string package. Relation to other approaches to resurgence in topological string theory? [Couso-Santamaria et al., Couso-Santamaria-M.M.-Schiappa]

# Conclusions and open questions

Resurgence can be used to extract precious information from perturbative series. In some cases, it leads to non-trivial integer invariants. This is a new route to integrality, different from previous ones (like radial asymptotics of $q$-series).

Our results determine (at least conjecturally) the **complete** resurgent structure of complex Chern-Simons theory for hyperbolic knots, and indicate a close relationship between Stokes constants and BPS counting in the "dual" 3d theory.

The same structures appear in topological string theory on toric CY manifolds, but it is not clear what is the enumerative meaning of the resulting integers. Determining the different sectors is also harder

**Many** open questions:

1) What happens if we turn on deformations of the hyperbolic structure? Do we find then a clearer relation to the A-polynomial and the AJ conjecture?

2) Can we obtain these results from a WKB analysis of the A-polynomial, a la GMN?

3) Can we reformulate the state integral invariants in terms of a Riemann-Hilbert problem?

4) Can we prove our conjectures or justify them physically?

5) Can we develop a similar theory for topological strings on toric CYs?

# Thank you for your attention!