Exponential integrals, Lefschetz thimbles and linear resurgence

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1. Exponential integrals (and why Borel summation works)

Given:

- X - complex manifold, $\dim_{\mathbb{C}} X = N$
- f - holomorphic function on X (a map $f : X \to \mathbb{C}$)
- $\boldsymbol{\mu}$ - holomorphic N-form on X, in local coordinates $\boldsymbol{\mu} = \mu(x)\, d^N x$
- γ - oriented noncompact N-cycle such that $\operatorname{Re} f_{|\gamma} \to +\infty$ at "infinity" of γ

$\rightsquigarrow$ integral depending on small parameter $0 < \hbar \ll 1$ (assume absolute convergence):

$$I(\hbar) := \int_\gamma e^{-f/\hbar} \boldsymbol{\mu} = \int_\gamma e^{-f(x)/\hbar} \mu(x)\, d^N x$$

Want to study its asymptotic expansion at $\hbar \to +0$. **Example**: $\displaystyle\int_{\mathbb{R}} e^{-\frac{1}{\hbar}(x^4+x)} dx$.

1. Exponential integrals (and why Borel summation works)

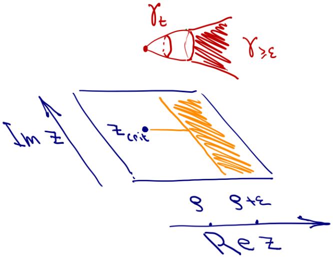
Define $\rho = \rho_{crit} \in \mathbb{R}$ as the maximal real number such that γ can be pushed to the subset

$$\{x \in X | \operatorname{Re} f(x) \geq \rho_{crit}\}$$

Then ρ_{crit} is a critical value of $\operatorname{Re} f$, hence equals to the real part of a complex critical value z_{crit} of f (or maybe several critical values $z_{crit,i}$).

Simplifying assumption: only one complex critical value, $\rightsquigarrow$ cycle γ can be pushed to

$$\Big(\bigcup_{t\in[0,\epsilon]} \gamma_t\Big) \cup \gamma_{\geq\epsilon}, \quad \gamma_t \in H_{N-1}(f^{-1}(z_{crit}+t), \mathbb{Z}), \quad \gamma_{\geq\epsilon} \subset \{x \in X | \operatorname{Re} f(x) \geq \rho_{crit} + \epsilon\}$$



$\lim_{t\to+0} \gamma_t = 0 \in H_{N-1}(f^{-1}(z_{crit}), \mathbb{Z})$
(vanishing cycle)

Contribution of $\gamma_{\geq\epsilon}$ is exponentially supressed.

1. Exponential integrals (and why Borel summation works)

Denote by $vol(t)$ the volume of $(N-1)$-dimensional cycle γ_t with respect to $(N-1)$- form $\frac{\boldsymbol{\mu}}{df}$ on $f^{-1}(z_{crit}+t)$, for $0 < t \le \epsilon$. Then we have

$$\int_\gamma e^{-f/\hbar}\boldsymbol{\mu} = e^{-z_{crit}/\hbar}\left(\int_0^\epsilon e^{-t/\hbar} vol(t)dt + O(e^{-\epsilon/\hbar})\right)$$

Theorem (follows from resolution of singularities): *Function* $vol(t)$ *for* $0 < t \ll \epsilon$ *is the sum of an absolutely convergent series:*

$$vol(t) = \sum_{\lambda\in\frac{1}{M}\{1,2,\dots\},0\le k\le k_{max}} a_{\lambda,k}\, t^{\lambda-1}\log(t)^k \qquad \textit{for some } M\ge 1,\ k_{max}\ge 0$$

Corollary: *Exponential integral has an asymptotic expansion at* $\hbar \to +0$

$$\int_\gamma e^{-f/\hbar}\boldsymbol{\mu} \sim e^{-z_{crit}/\hbar}\cdot \sum_{\substack{\lambda\in\frac{1}{M}\{1,2,\dots\}\\ 0\le k\le k_{max}}} c_{\lambda,k}\,\hbar^\lambda \log(\hbar)^k$$

1. Exponential integrals (and why Borel summation works)

Simplifying assumption: $k_{max} = 0$. Then $vol(t) = \sum_{\lambda>0} a_\lambda \, t^{\lambda-1}$ and

$$e^{z_{crit}/\hbar} \int_\gamma e^{-f/\hbar} \boldsymbol{\mu} \sim \sum_{\lambda>0} c_\lambda \, \hbar^\lambda \quad \text{where } c_\lambda = \Gamma(\lambda)\, a_\lambda \text{ because } \int_0^\infty e^{-t/\hbar} \, t^{\lambda-1} dt = \Gamma(\lambda)\, \hbar^\lambda$$

Asymptotic series $\sum_\lambda c_\lambda \, \hbar^\lambda$ is *factorially divergent*. How do we get a numerical value?

Borel summation method: Apply *Borel transform*

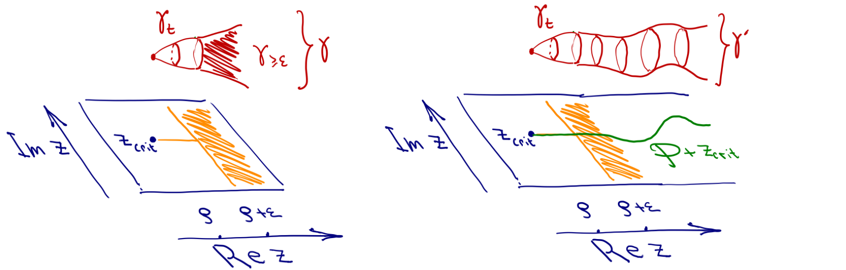
$$\sum_\lambda c_\lambda \, \hbar^\lambda \rightsquigarrow \sum_\lambda \frac{c_\lambda}{\Gamma(\lambda)} t^{\lambda-1} \quad \left(\text{and get } \sum_\lambda a_\lambda \, t^{\lambda-1} = vol(t) \right)$$

Assume that the Borel transform extends analytically along a path $\mathcal{P}$ in $\mathbb{C}$ starting at 0 and going to $+\infty$ (which is in our case any path avoiding critical values of $f - z_{crit}$).

$$\text{Regularized value of } \sum_\lambda c_\lambda \, \hbar^\lambda := \int_{\mathcal{P}} e^{-t/\hbar} \text{ Borel transform}(t) dt$$

1. Exponential integrals (and why Borel summation works)

Choice of a path $\mathcal{P} \rightsquigarrow$ a cycle of integration γ' obtained as the union of $(N-1)$-dimensional cycles $\gamma_t \in H_{N-1}(f^{-1}(z_{crit}+t), \mathbb{Z})$ for $t \in \mathcal{P}$.



The difference $\gamma - \gamma'$ is a cycle of integration supported *strictly on the right*, in the subset $\{x \in X | \operatorname{Re} f(x) > \rho_{crit} + \epsilon\} \subset X$. In a lucky case, when, e.g. there is no critical values of f with real part $> \rho_{crit}$, we can push $\gamma - \gamma'$ further to $+\infty$ and show that the integral over $\gamma - \gamma'$ vanishes, i.e. Borel summation gives exactly the value of $e^{z_{crit}/\hbar} \int_\gamma e^{-f/\hbar} \boldsymbol{\mu}$. Otherwise, we get a correction with a strictly faster exponential decay, which can be analyzed in a similar way.

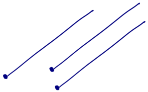
2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

Simplifying assumption: all critical points $x_\alpha \in X$ are *isolated* and *non-degenerate* (i.e. Morse in the holomorphic sense: $f''_{|x_\alpha}$ is a non-degenerate $\mathbb{C}$-quadratic form on $T_{x_\alpha}X$), and all critical values $z_\alpha = f(x_\alpha)$ are *pairwise distinct*.
There is a unique (up to $\pm$) vanishing cycle near each x_α, diffeomorphic to S^{N-1}.

Definition: for each Morse critical point x_α and a *generic* direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the **Lefschetz thimble** $th_{\alpha,\theta} \simeq \mathbb{R}^N \subset X$ is the continuation of the vanishing cycle near x_α along the path $z_\alpha + e^{i\theta}\mathbb{R}_{\geq 0} \subset \mathbb{C}$. Ill-defined only if $\theta = \arg(z_\beta - z_\alpha)$ for some $z_\beta \neq z_\alpha$.

For a *complex* value $\hbar \in \mathbb{C} - 0$ and a critical point x_α, the **normalized integral** is

$$I_\alpha^{norm}(\hbar) := \frac{1}{(2\pi\hbar)^{N/2}} e^{z_\alpha/\hbar} \int_{th_{\alpha,\arg\hbar}} e^{-f/\hbar} \boldsymbol{\mu}$$

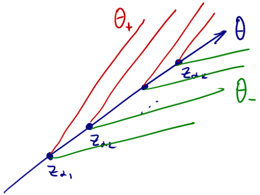


By general theory, it has a divergent asymptotic expansion $\sim \sum_{n\geq 0} c_n^{(\alpha)}\,\hbar^n$.
Coefficients $c_0^{(\alpha)}, c_1^{(\alpha)}, \ldots$ *do not depend* on the direction $\theta = \arg(\hbar)$.

2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

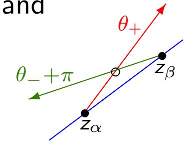
Function I_α^{norm} is defined and *analytic* outside Stokes rays $\{\hbar \,|\, \arg \hbar = \arg(z_\beta - z_\alpha)\}$ in the complex plane $\mathbb{C}_\hbar$, extends analytically a little bit across the cuts.
Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be such that on an oriented line in $\mathbb{C}$ with slope θ lie $k \geq 2$ distinct critical values $z_{\alpha_1}, \dots, z_{\alpha_k}$ in the increasing order. For values $\theta_- < \theta < \theta_+$ close to θ the normalized integrals are well-defined.

Jump formula :
$$\begin{pmatrix} I_{\alpha_1,+}^{norm}(\hbar) \\ \vdots \\ I_{\alpha_k,+}^{norm}(\hbar) \end{pmatrix} = S_\theta \cdot \begin{pmatrix} I_{\alpha_1,-}^{norm}(\hbar) \\ \vdots \\ I_{\alpha_k,-}^{norm}(\hbar) \end{pmatrix}$$



where $S_\theta = (S_{\alpha,\beta})_{\alpha,\beta \in \{\alpha_1,\dots,\alpha_k\}}$ is an upper-triangular matrix with $S_{\alpha,\alpha} = 1$ and exponentially small off-diagonal terms $S_{\alpha,\beta} = S_{\alpha,\beta}(\hbar) := n_{\alpha,\beta}\, e^{-(z_\beta - z_\alpha)/\hbar}$ where $n_{\alpha,\beta}$ are *integers* (Stokes indices) given by intersection numbers

$$n_{\alpha,\beta} := th_{\alpha,\theta_+} \cdot th_{\beta,\theta_- + \pi} \in \mathbb{Z}$$



2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

Data $(z_\alpha \in \mathbb{C}), (n_{\alpha,\beta} \in \mathbb{Z})$ defines a modification of the trivial vector bundle with fiber $\mathbb{C}^{\{z_\alpha\}}$ over complex plane $\mathbb{C}_\hbar$, by gluing via linear[1] automorphisms $S_\theta(\hbar)$ along Stokes rays $e^{i\theta}\mathbb{R}_{\geq 0}$. Let solve somehow (there are many ways) *Riemann-Hilbert problem*, i.e. find a holomorphic trivialization of the glued bundle. Equivalently, find matrix-valued function G on $\mathbb{C}_\hbar - \{\text{Stokes rays}\}$ satisfying jump equations on cuts

$$G_{e^{i\theta},+}(\hbar) = S_\theta(\hbar) \cdot G_{e^{i\theta},-}(\hbar), \qquad \text{with boundary condition: } \lim_{\hbar \to 0} G(\hbar) = Id_{\mathbb{C}^{\{z_\alpha\}}}$$

Then vector-valued function $\vec{J}(\hbar) := G(\hbar)^{-1} \cdot \vec{I}^{norm}(\hbar)$ has *trivial* jumps, hence is *holomorphic* in $\hbar$. The convergent Taylor expansion of $\vec{J}$ is the product of the divergent asymptotic expansion of G at $\hbar = 0$ (does not depend on the choice of sector), and of the divergent asymptotic expansion of $\vec{I}^{norm}(\hbar) = (I_\alpha^{norm}(\hbar))_{all\{z_\alpha\}}$.

$\boxed{\vec{I}^{norm} = G \cdot J}$ - an alternative to Borel summation: divergent series $\rightsquigarrow$ actual values.

[1]This is the origin of "linear resurgence" in the title. WKB problems lead to *non-linear* glueings.

3. Example: classical special functions

- (Airy function): $X=\mathbb{C}$, $f=\frac{x^3}{3}-x$, $\boldsymbol{\mu}=dx$. Critical points: $x=\pm 1$, critical values: $z_1=-2/3, z_2=+2/3$. Stokes indices: $n_{12}=+\mathbf{1}$, $n_{21}=-\mathbf{1}$.

z_1 z_2

For $\hbar \in \mathbb{C}-\mathbb{R}_{\leq 0}$: $I_2^{norm}=\sqrt{2\pi}\frac{e^{2/(3\hbar)}}{\hbar^{1/6}}\,\mathrm{Ai}\left(\hbar^{-2/3}\right)\sim\frac{1}{\sqrt{2}}\sum_{n\geq 0}\frac{(6n-1)!!}{(2n)!\,72^n}\hbar^n$

For $\hbar \in \mathbb{C}-\mathbb{R}_{\geq 0}$: $I_1^{norm}(\hbar)=-iI_2^{norm}(-\hbar)$

where $\mathrm{Ai}(y):=1/(2\pi)\int_{\mathbb{R}}\cos(t^3/3+yt)dt$ is the classical Airy function.

Jumps:

$$I_1^{norm}(\hbar+i0)-I_1^{norm}(\hbar-i0)=\ +\mathbf{1}\cdot e^{-\frac{4}{3\hbar}}I_2^{norm}(\hbar),\quad \hbar\in\mathbb{R}_{\geq 0}$$

$$I_2^{norm}(\hbar-i0)-I_2^{norm}(\hbar+i0)=\ -\mathbf{1}\cdot e^{+\frac{4}{3\hbar}}I_1^{norm}(\hbar),\quad \hbar\in\mathbb{R}_{\leq 0}$$

modified, of second kind

- (Bessel function): $X=\mathbb{C}^\times$, $f=x+\frac{1}{x}$, $\boldsymbol{\mu}=\frac{dx}{x}$. Critical points: $x=\pm 1$, critical values: $z_1=-2, z_2=+2$. Stokes indices: $n_{12}=+\mathbf{2}$, $n_{21}=-\mathbf{2}$.

Bessel integral: $\int_0^\infty e^{-\frac{1}{\hbar}(x+\frac{1}{x})}\frac{dx}{x}=2\mathrm{K}_0\left(\frac{2}{\hbar}\right)\sim\sqrt{\pi\hbar}\,e^{-\frac{2}{\hbar}}\sum_{n\geq 0}\frac{(-1)^n((2n-1)!!)^2}{n!\,16^n}\hbar^n$

4. Example: Gamma function, infinitely many critical values

Consider algebraic variety $\mathbb{C}^\times$ with (closed) 1-form $\eta = (1-1/x)dx$ which is *not* a differential of a function, as it has a non-trivial period $\oint \eta = -2\pi i$. In order to represent η as differential of a function, we should go to the universal $\mathbb{Z}$-cover $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$. Denote by t the coordinate on the cover, so $x = e^t$. Then we have

$$(\text{pullback of } \eta) = dx - \frac{dx}{x} = dx - dt = d(e^t - t), \quad f := e^t - t$$

$\vdots$
$\bullet\, z_{-1}$
$\bullet\, z_0$
$\bullet\, z_1$
$\bullet\, z_2$
$\vdots$

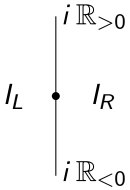
Function $f = f(t)$ has *infinitely many* critical points $t_k = 2\pi i k, \quad k \in \mathbb{Z}$.
$\rightsquigarrow$ critical values: $z_k = 1 - 2\pi i k$. The normalized integral for $\operatorname{Re}\hbar > 0$, *any* $k \in \mathbb{Z}$ is

$$I_k^{norm}(\hbar) \overset{k=0}{:=} \frac{e^{1/\hbar}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{(-e^t+t)/\hbar} dt = \frac{e^{1/\hbar}}{\sqrt{2\pi\hbar}} \int_0^{\infty} e^{-x/\hbar} x^{1/\hbar} \frac{dx}{x} = \frac{e^{1/\hbar}\hbar^{1/\hbar}\,\Gamma(1/\hbar)}{\sqrt{2\pi\hbar}}$$

For $\operatorname{Re}\hbar < 0$ one has $I_\bullet^{norm}(\hbar) = 1/I_\bullet^{norm}(-\hbar)$ (does not depend on $\bullet = k \in \mathbb{Z}$)

Asymptotic expansion: $I_\bullet^{norm}(\hbar) \overset{\hbar\to+0}{\sim} 1 + \frac{1}{12}\hbar + \frac{1}{288}\hbar^2 - \frac{139}{51840}\hbar^3 + \ldots$ (*Stirling formula*).

4. Example: Gamma function, infinitely many critical values



We get basically *two* functions: $\begin{cases} I_R(\hbar) & := \frac{e^{1/\hbar}\hbar^{1/\hbar}\,\Gamma(1/\hbar)}{\sqrt{2\pi\hbar}} & \text{for } \operatorname{Re}\hbar > 0 \\ I_L(\hbar) & := 1/I_R(-\hbar) & \text{for } \operatorname{Re}\hbar < 0 \end{cases}$

Jump formulas:

$$I_L(\hbar) = I_R(\hbar)\cdot\left(1-\exp\left(-\tfrac{2\pi i}{\hbar}\right)\right) \quad \text{for } \hbar \in i\,\mathbb{R}_{>0}$$
$$I_R(\hbar) = I_L(\hbar)\cdot\left(1-\exp\left(+\tfrac{2\pi i}{\hbar}\right)\right)^{-1} \quad \text{for } \hbar \in i\,\mathbb{R}_{<0}$$

$$\rightsquigarrow \text{ Stokes indices } n_{kk'} = \begin{cases} -1 & k' = k-1 \\ +1 & k' > k \\ 0 & \text{otherwise} \end{cases} \qquad \forall k, k' \in \mathbb{Z}$$

Puzzle: for quantized values of $\hbar = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$ function $e^{-f/\hbar}$ *descends* to $\mathbb{C}^\times$-valued function on $\mathbb{Z}$-quotient $\mathbb{C}^\times$ with coordinate $x = e^t \neq 0$. Hence for such $\hbar$ the contour integral $\oint e^{-f/\hbar}dx/x$ is well-defined. How it is related to the integrals over Lefschetz thimbles? A similar problem in quantum Chern-Simons theory, see later.

5. Example: Infinite-dimensional integral (heat kernel)

Let (M, g) be a real-analytic Riemannian manifold which admits a "reasonable" complexification $(M_{\mathbb{C}}, g_{\mathbb{C}})$. Fix two points $p_0, p_1 \in M$, and consider the *path integral*

$$I(\hbar) := \int_{\substack{\text{Paths } \phi:[0,1]\to M \\ \phi(0)=p_0, \phi(1)=p_1}} e^{-\frac{S(\phi)}{\hbar}} \mathcal{D}\phi, \qquad \text{where } S(\phi) = \frac{1}{2}\int_0^1 \left|\frac{d\phi(t)}{dt}\right|_g^2 dt$$

The integration domain is as a totally real infinite-dimensional contour γ in $X :=$ infinite-dimensional *complex* manifold of paths connecting p_0 and p_1 in $M_{\mathbb{C}}$.
Dirichlet functional $\phi \in \gamma \mapsto S(\phi) \in \mathbb{R}$ extends to a *holomorphic* function $f : X \to \mathbb{C}$,
the fictitious "Lebesgue measure" $\mathcal{D}\phi$ "extends" to a holomorphic volume form $\boldsymbol{\mu}$ on X.

Mathematically rigorous interpretation (Feynman-Kac formula):

$$I(\hbar) = \text{ heat kernel } K(p_0, p_1; t) = \langle p_0 | e^{-\frac{1}{2}\Delta t} | p_1 \rangle$$

where *time* $t > 0$ (for the Brownian motion) is identified $\overset{t:=\hbar}{\text{}}$ with the *Planck constant* $\hbar$.

5. Example: Infinite-dimensional integral (heat kernel)

Short-time asymptotic expansion of the heat kernel:

$$K(p_0, p_1; t) \overset{t\to+0}{\sim} (2\pi t)^{-\dim M/2}\, e^{-\frac{dist(p_0,p_1)^2}{2t}} \sum_{n\geq 0} c_n t^n$$

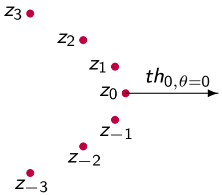
Conjecture: sequence c_n has factorial growth, is resurgent, and the Borel transform has singularities at numbers $\ell_\alpha^2/2 \in \mathbb{C}$ where ℓ_α are lengths of *complex* geodesics connecting p_0, p_1 (formal solutions of Euler-Lagrange equation for $\phi : [0,1] \to (M_{\mathbb{C}}, g_{\mathbb{C}})$).

McKean, 1970

$X = \mathbb{H}^2$:

$$K(p_0, p_1; t) = \frac{\sqrt{2}\, e^{-t/8}}{(2\pi t)^{3/2}} \int_{\ell^2/2}^{\infty} \frac{e^{-z/t}}{(\cosh\sqrt{2z} - \cosh\ell)^{1/2}} dz \qquad \ell := dist(p_0, p_1)$$

hyperelliptic curve of infinite genus!



Critical values

$$z_n = \frac{(\ell + 2\pi i n)^2}{2}$$

Also works for spheres, flat tori, compact surfaces with hyperbolic metric,. . .

6. Example: complex Chern-Simons, paths in K_2-geometry, sums of dilogs

3 different series of examples, hypothetically give the *same* class of data $(z_\alpha), (n_{\alpha,\beta})$:

- $X_3 = \mathbb{Z}$-cover of $X_3' := \left\{ C^\infty\text{-connections on G-bundle } \begin{matrix} \mathrm{P} \\ \downarrow \\ \mathrm{M} \end{matrix} \right\} /\text{gauge equivalences},$

  where M is an oriented manifold of dimension 3, G is a complex semisimple group.

- $X_1 = \mathbb{Z}$-cover of $X_1' := \left\{ C^\infty\text{-paths } \phi : [0,1] \to (\mathbb{C}^\times)^{2n} \,|\, \phi(0) \in L_0, \phi(1) \in L_1 \right\}$

  where L_0, L_1 are algebraic K_2-lagrangian submanifolds of $(\mathbb{C}^\times)^{2n}$:

  $\sum_{i=1}^{n} [z_i]_{|L_\epsilon} \wedge [z_{n+i}]_{|L_\epsilon} = 0 \in K_2(L_\epsilon), \quad \epsilon = 0, 1 \quad (K_2\text{-condition will explain later})$

- $X_0 = \mathbb{Z}$-cover of $X_0' :=$ a $\mathbb{Z}^m$-cover of $(\mathbb{C}^\times)^m - \bigcup_{\vec{\nu} \in B} \{\vec{x} \in (\mathbb{C}^\times)^m \mid \vec{x}^{\vec{\nu}} = 1\}$

  where $B \subset \mathbb{Z}^m - 0$ is a finite subset,

  $\vec{x}^{\vec{\nu}} := \prod_{i=1}^{m} x_i^{\nu_i}$ for $\vec{x} = (x_1, \ldots, x_m) \in (\mathbb{C}^\times)^m$ and $\vec{\nu} = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m$.

6. Example: complex Chern-Simons, paths in K_2-geometry, sums of dilogs

Case of connections on a 3-dimensional manifold M:

On $X_3 = \{Connections\}$ functional f is the *Chern-Simons action*:
for a $SL(N, \mathbb{C})$-connection $\nabla = d + A$ in the trivialized bundle

$$f(\nabla) := \int_{\mathrm{M}} \mathrm{Tr}\left(\frac{AdA}{2} + \frac{A^3}{3}\right), \qquad A \in Mat(N \times N, \Omega^1(\mathrm{M})),\ \mathrm{Tr}(A) = 0$$

Ambiguity under gauge transformations: $f(g^{-1}\nabla g) - f(\nabla) \in (2\pi i)^2\mathbb{Z}$.

For special "quantized" values of $\hbar$:

$$\hbar = \frac{2\pi i}{k},\ k = \pm 1, \pm 2, \ldots$$

$\rightsquigarrow$ well-defined function $\exp(-f/\hbar)$ on $\mathbb{Z}$-quotient X_3'. Canonical "compact" cycle of integration $= \{$unitary connections$\}$ $\rightsquigarrow$ quantum Chern-Simons theory at level k (for $k > 0$).

6. Example: complex Chern-Simons, paths in K_2-geometry, sums of dilogs

Case of paths connecting K_2-lagrangian subvarieties $L_0, L_1 \in (\mathbb{C}^\times)^{2n}$:

Examples of K_2-lagrangian subvarieties in $(\mathbb{C}^\times)^{2n}$ endowed with the standard symplectic form $\omega = \sum_{i=1}^n d\log(z_i) \wedge d\log(z_{n+i})$:

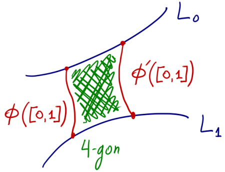
$$L = \{(z_1, \ldots, z_{2n}) \in (\mathbb{C}^\times)^{2n} \,|\, \forall\, i = 1, \ldots, n: \quad z_{n+i} = 1 \text{ or } (1 - z_i)\}$$

and its images under $Sp(2n, \mathbb{Z})$-action.

For any K_2-lagrangian L and any $\delta \in H_2((\mathbb{C}^\times)^{2n}, L; \mathbb{Z})$ one has $\int_\delta \omega \in (2\pi i)^2\mathbb{Z}$.

Functional f on $\{$paths connecting two K_2-lagrangian $L_0, L_1\}$ is defined up to $(2\pi i)^2\mathbb{Z}$:

$$f(\phi) - f(\phi') = \int_{4-gon} \omega$$



6. Example: complex Chern-Simons, paths in K_2-geometry, sums of dilogs

Case of sums of dilogarithms:

DATA:

- a finite subset $B \subset \mathbb{Z}^m - 0$
- a collection of integer non-zero weights $w_{\vec{\nu}} \in \mathbb{Z} - 0$ for all $\vec{\nu} \in B$
- an even quadratic form $b = (b_{ij})_{1 \leq i,j \leq m}$, $\quad b_{ij} = b_{ji} \in \mathbb{Z}, b_{ii} \in 2\mathbb{Z}$

$\rightsquigarrow$ multivalued function

$$f := \sum_{\vec{\nu} \in B} w_{\vec{\nu}} \, \mathrm{Li}_2(\vec{x}^{\vec{\nu}}) + \frac{1}{2} \sum_{i,j} b_{ij} \log(x_i) \log(x_j) \qquad \text{recall: } \mathrm{Li}_2(x) := \sum_{k \geq 1} \frac{x^k}{k^2}$$

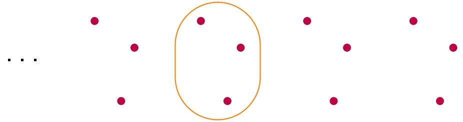
Its differential $\eta := df$ is well-defined on $X' := \mathbb{Z}^m$-cover of $(\mathbb{C}^\times)^m - \cup_{\vec{\nu} \in B} \{\vec{x}^{\vec{\nu}} = 1\}$

$$df = \sum_{i=1}^{m} \left(- \sum_{\vec{\nu} \in B} \nu_i w_{\vec{\nu}} \log(1 - \vec{x}^{\vec{\nu}}) + \sum_{j=1}^{m} b_{ij} \log(z_j) \right) d \log(x_i)$$

Periods of η belong to $(2\pi i)^2 \mathbb{Z}$, function f is well-defined on a $\mathbb{Z}$-cover X of X'.

6. Example: complex Chern-Simons, paths in K_2-geometry, sums of dilogs

In all 3 situations the set of critical values is a *finite* union of arithmetic progressions in $\mathbb{C}$ each with the step $(2\pi i)^2 = -39.4784\ldots$.



$$\alpha = (\underline{\alpha}, k)$$
$$z_\alpha = z_{\underline{\alpha}} + (2\pi i)^2 k$$

The reason is that in each of 3 situations, critical points (up to $\mathbb{Z}$-action) are solutions of a system of algebraic equations, with coefficients in $\mathbb{Q}$, of expected dimension 0.

- case **3**: representations $\pi_1(\mathrm{M}) \to \mathrm{G}$ (the group can be defined over $\mathbb{Q}$)
- case **1**: intersection $L \cap L'$ of two algebraic subvarieties defined over $\mathbb{Q}$
- case **0**: solutions of a system of *algebraic* equations $\exp(x_i \partial_{x_i} f) = 1$, $i = 1, \ldots, m$

Same critical values, - image of Beilinson-Borel regulator $K_3^{ind}(\overline{\mathbb{Q}}) \to \mathbb{C}/(2\pi i\mathbb{Z})^2$.

6. Example: complex Chern-Simons, paths in K_2-geometry, sums of dilogs

Conjecture: one can identify situations **3**,**1**,**0** not only matching the critical values $z_{(\underline{\alpha},k)}$, but also the *Stokes indices* $n_{(\underline{\alpha},k),(\underline{\alpha}',k')} =: n_{\underline{\alpha},\underline{\alpha}';k'-k}$.

Analogy: Stokes indices of an ∞-dim. path integral (heat kernel) = those of a finite-dim. exponential integral.

ad hoc

One can effectively study topology in the case **0**, an example:

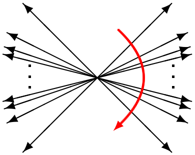
$$f = \mathrm{Li}_2(x) + \log(x)^2 \rightsquigarrow X_0' = \{(x,t) \in \mathbb{C}^2 | \tfrac{x^2}{1-x} = e^t\} \Leftrightarrow x = \underbrace{\frac{-e^t \pm e^{t/2}\sqrt{e^t+4}}{2}}$$

∞ genus hyperelliptic curve

Map $\exp(\frac{f}{2\pi i}) : X_0' \to \mathbb{C}^\times$ is ramified at a finite set of points in $\mathbb{C}^\times$, monodromy is accessible.

The angle-ordered product $\overset{\curvearrowright}{\prod}$ of ∞ many Stokes matrices S_θ can be identified with the "monodromy" of a q-difference equation where

$$q = e^{\frac{(2\pi i)^2}{\hbar}}$$



replace by one cut in RH problem