Exponential integrals, Lefschetz thimbles and linear resurgence

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1. Exponential integrals (and why Borel summation works)

Given:

- $X$ - complex manifold, $\dim_{\mathbb{C}} X = N$
- $f$ - holomorphic function on $X$ (a map $f : X \to \mathbb{C}$)
- $\mu$ - holomorphic $N$-form on $X$, in local coordinates $\mu = \mu(x) \, d^N x$
- $\gamma$ - oriented noncompact $N$-cycle such that $\operatorname{Re} f|_{\gamma} \to +\infty$ at “infinity” of $\gamma$

$\sim$ integral depending on small parameter $0 < \hbar \ll 1$ (assume absolute convergence):

$$ I(\hbar) := \int_{\gamma} e^{-f(x)/\hbar} \mu = \int_{\gamma} e^{-f(x)/\hbar} \mu(x) \, d^N x $$

Want to study its asymptotic expansion at $\hbar \to +0$. Example: $\int_{\mathbb{R}} e^{-\frac{1}{\hbar}(x^4+x)} \, dx$. 
1. Exponential integrals (and why Borel summation works)

Define $\rho = \rho_{\text{crit}} \in \mathbb{R}$ as the maximal real number such that $\gamma$ can be pushed to the subset

$$\{x \in X \mid \Re f(x) \geq \rho_{\text{crit}}\}$$

Then $\rho_{\text{crit}}$ is a critical value of $Re f$, hence equals to the real part of a complex critical value $z_{\text{crit}}$ of $f$ (or maybe several critical values $z_{\text{crit},i}$).

**Simplifying assumption:** only one complex critical value, $\leadsto$ cycle $\gamma$ can be pushed to

$$\left( \bigcup_{t \in [0, \epsilon]} \gamma_t \right) \cup \gamma_{\geq \epsilon}, \quad \gamma_t \in H_{N-1}(f^{-1}(z_{\text{crit}} + t), \mathbb{Z}), \quad \gamma_{\geq \epsilon} \subset \{x \in X \mid \Re f(x) \geq \rho_{\text{crit}} + \epsilon\}$$

$$\lim_{t \to +0} \gamma_t = 0 \in H_{N-1}(f^{-1}(z_{\text{crit}}), \mathbb{Z})$$

(\textbf{vanishing cycle})

Contribution of $\gamma_{\geq \epsilon}$ is exponentially suppressed.
Denote by $\text{vol}(t)$ the volume of $(N-1)$-dimensional cycle $\gamma_t$ with respect to $(N-1)$- form $\frac{\mu}{df}$ on $f^{-1}(z_{\text{crit}} + t)$, for $0 < t \leq \epsilon$. Then we have

$$\int_{\gamma} e^{-f/\hbar} \mu = e^{-z_{\text{crit}}/\hbar} \left(\int_{0}^{\epsilon} e^{-t/\hbar} \text{vol}(t) dt + O(e^{-\epsilon/\hbar})\right)$$

**Theorem** (follows from resolution of singularities): *Function vol(t) for $0 < t \ll \epsilon$ is the sum of an absolutely convergent series:*

$$\text{vol}(t) = \sum_{\lambda \in \frac{1}{M}\{1,2,\ldots\},0 \leq k \leq k_{\text{max}}} a_{\lambda,k} t^{\lambda-1} \log(t)^k \quad \text{for some } M \geq 1, k_{\text{max}} \geq 0$$

**Corollary:** *Exponential integral has an asymptotic expansion at $\hbar \rightarrow +0$*

$$\int_{\gamma} e^{-f/\hbar} \mu \sim e^{-z_{\text{crit}}/\hbar} \cdot \sum_{\lambda \in \frac{1}{M}\{1,2,\ldots\}, 0 \leq k \leq k_{\text{max}}} c_{\lambda,k} \hbar^\lambda \log(\hbar)^k$$
1. Exponential integrals (and why Borel summation works)

Simplifying assumption: $k_{max} = 0$. Then \( \text{vol}(t) = \sum_{\lambda > 0} a_{\lambda} t^{\lambda - 1} \) and

\[
e^{z_{\text{crit}}/\hbar} \int_{\gamma} e^{-f/\hbar} \mu \sim \sum_{\lambda > 0} c_{\lambda} \hbar^\lambda \quad \text{where } c_{\lambda} = \Gamma(\lambda) a_{\lambda} \text{ because } \int_0^\infty e^{-t/\hbar} t^{\lambda - 1} dt = \Gamma(\lambda) \hbar^\lambda
\]

Asymptotic series $\sum_{\lambda} c_{\lambda} \hbar^\lambda$ is factorially divergent. How do we get a numerical value?

**Borel summation method:** Apply Borel transform

\[
\sum_{\lambda} c_{\lambda} \hbar^\lambda \leadsto \sum_{\lambda} \frac{c_{\lambda}}{\Gamma(\lambda)} t^{\lambda - 1} \quad \left( \text{and get } \sum_{\lambda} a_{\lambda} t^{\lambda - 1} = \text{vol}(t) \right)
\]

Assume that the Borel transform extends analytically along a path $P$ in $\mathbb{C}$ starting at 0 and going to $+\infty$ (which is in our case any path avoiding critical values of $f - z_{\text{crit}}$).

Regularized value of $\sum_{\lambda} c_{\lambda} \hbar^\lambda := \int_P e^{-t/\hbar} \text{Borel transform}(t) dt$
1. Exponential integrals (and why Borel summation works)

Choice of a path $\mathcal{P} \leadsto$ a cycle of integration $\gamma'$ obtained as the union of $(N - 1)$-dimensional cycles $\gamma_t \in H_{N-1}(f^{-1}(z_{\text{crit}} + t), \mathbb{Z})$ for $t \in \mathcal{P}$.

The difference $\gamma - \gamma'$ is a cycle of integration supported strictly on the right, in the subset $\{x \in X | \text{Re } f(x) > \rho_{\text{crit}} + \epsilon\} \subset X$. In a lucky case, when, e.g. there is no critical values of $f$ with real part $> \rho_{\text{crit}}$, we can push $\gamma - \gamma'$ further to $+\infty$ and show that the integral over $\gamma - \gamma'$ vanishes, i.e. Borel summation gives exactly the value of $e^{z_{\text{crit}}/\hbar} \int_{\gamma} e^{-f/\hbar} \mu$. Otherwise, we get a correction with a strictly faster exponential decay, which can be analyzed in a similar way.
2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

Simplifying assumption: all critical points $x_\alpha \in X$ are isolated and non-degenerate (i.e. Morse in the holomorphic sense: $f''_{x_\alpha}$ is a non-degenerate $\mathbb{C}$-quadratic form on $T_{x_\alpha}X$), and all critical values $z_\alpha = f(x_\alpha)$ are pairwise distinct.

There is a unique (up to $\pm$) vanishing cycle near each $x_\alpha$, diffeomorphic to $S^{N-1}$.

**Definition:** for each Morse critical point $x_\alpha$ and a generic direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the Lefschetz thimble $th_{\alpha,\theta} \simeq \mathbb{R}^N \subset X$ is the continuation of the vanishing cycle near $x_\alpha$ along the path $z_\alpha + e^{i\theta} \mathbb{R}_{\geq 0} \subset \mathbb{C}$. Ill-defined only if $\theta = \text{arg}(z_\beta - z_\alpha)$ for some $z_\beta \neq z_\alpha$.

For a complex value $\hbar \in \mathbb{C} - \{0\}$ and a critical point $x_\alpha$, the normalized integral is

$$I_{\alpha}^{\text{norm}}(\hbar) := \frac{1}{(2\pi \hbar)^{N/2}} e^{z_\alpha/\hbar} \int_{th_{\alpha,\arg \hbar}} e^{-f/\hbar} \mu$$

By general theory, it has a divergent asymptotic expansion $\sim \sum_{n \geq 0} c_n^{(\alpha)} \hbar^n$. Coefficients $c_0^{(\alpha)}, c_1^{(\alpha)}, \ldots$ do not depend on the direction $\theta = \text{arg}(\hbar)$. 
2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

Function $I_{\alpha}^{\text{norm}}$ is defined and \textit{analytic} outside Stokes rays \{\(\hbar \mid \arg \hbar = \arg(z_\beta - z_\alpha)\}\) in the complex plane \(\mathbb{C}_\hbar\), extends analytically a little bit across the cuts.

Let \(\theta \in \mathbb{R}/2\pi\mathbb{Z}\) be such that on an oriented line in \(\mathbb{C}\) with slope \(\theta\) lie \(k \geq 2\) distinct critical values \(z_{\alpha_1}, \ldots, z_{\alpha_k}\) in the increasing order. For values \(\theta_- < \theta < \theta_+\) close to \(\theta\) the normalized integrals are well-defined.

\textbf{Jump formula}:

\[
\begin{pmatrix}
I_{\alpha_1, +}^{\text{norm}}(\hbar) \\
\vdots \\
I_{\alpha_k, +}^{\text{norm}}(\hbar)
\end{pmatrix} = S_\theta \cdot
\begin{pmatrix}
I_{\alpha_1, -}^{\text{norm}}(\hbar) \\
\vdots \\
I_{\alpha_k, -}^{\text{norm}}(\hbar)
\end{pmatrix}
\]

where \(S_\theta = (S_{\alpha, \beta})_{\alpha, \beta \in \{\alpha_1, \ldots, \alpha_k\}}\) is an upper-triangular matrix with \(S_{\alpha, \alpha} = 1\) and exponentially small off-diagonal terms \(S_{\alpha, \beta}(\hbar) := n_{\alpha, \beta} \cdot e^{-(z_\beta - z_\alpha)/\hbar}\) where \(n_{\alpha, \beta}\) are \textit{integers} (Stokes indices) given by intersection numbers

\[
n_{\alpha, \beta} := th_{\alpha, \theta_+} \cdot th_{\beta, \theta_- + \pi} \in \mathbb{Z}
\]
2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

Data \((z_\alpha \in \mathbb{C}), (n_\alpha, \beta \in \mathbb{Z})\) defines a modification of the trivial vector bundle with fiber \(\mathbb{C}\{z_\alpha\}\) over complex plane \(\mathbb{C}_\hbar\), by gluing via linear automorphisms \(S_\theta(\hbar)\) along Stokes rays \(e^{i\theta} \mathbb{R}_{\geq 0}\). Let solve somehow (there are many ways) Riemann-Hilbert problem, i.e. find a holomorphic trivialization of the glued bundle. Equivalently, find matrix-valued function \(G\) on \(\mathbb{C}_\hbar - \{\text{Stokes rays}\}\) satisfying jump equations on cuts

\[
G_{e^{i\theta},+}(\hbar) = S_\theta(\hbar) \cdot G_{e^{i\theta},-}(\hbar), \quad \text{with boundary condition: } \lim_{\hbar \to 0} G(\hbar) = Id_{\mathbb{C}\{z_\alpha\}}
\]

Then vector-valued function \(\vec{J}(\hbar) := G(\hbar)^{-1} \cdot \vec{I}_{\text{norm}}(\hbar)\) has trivial jumps, hence is holomorphic in \(\hbar\). The convergent Taylor expansion of \(\vec{J}\) is the product of the divergent asymptotic expansion of \(G\) at \(\hbar = 0\) (does not depend on the choice of sector), and of the divergent asymptotic expansion of \(\vec{I}_{\text{norm}}(\hbar) = (I_{\alpha, \text{norm}}(\hbar))_{\alpha \in \{z_\alpha\}}\).

\[
\vec{I}_{\text{norm}} = G \cdot J
\] - an alternative to Borel summation: divergent series \(\rightsquigarrow\) actual values.

\(^1\)This is the origin of “linear resurgence” in the title. WKB problems lead to non-linear glueings.
3. Example: classical special functions

(Airy function): \( X = \mathbb{C}, f = \frac{x^3}{3} - x, \mu = dx \). Critical points: \( x = \pm 1 \),
critical values: \( z_1 = -2/3, z_2 = +2/3 \). Stokes indices: \( n_{12} = +1, n_{21} = -1 \).

\[
\begin{align*}
\dot{z}_1 & \quad \dot{z}_2 \\
\text{For } \hbar \in \mathbb{C} - \mathbb{R}_{\leq 0} : & \quad I_2^{\text{norm}} = \sqrt{2\pi} \frac{e^{2/(3\hbar)}}{\hbar^{1/6}} \text{Ai}(\hbar^{-2/3}) \sim \frac{1}{\sqrt{2}} \sum_{n\geq 0} \frac{(6n - 1)!!}{(2n)! 72^n} \hbar^n \\
\text{For } \hbar \in \mathbb{C} - \mathbb{R}_{\geq 0} : & \quad I_1^{\text{norm}}(\hbar) = -iI_2^{\text{norm}}(-\hbar)
\end{align*}
\]

where \( \text{Ai}(y) := \int_{\mathbb{R}} \cos(t^{3/2} + yt)\,dt \) is the classical Airy function.

\[
I_1^{\text{norm}}(\hbar + i0) - I_1^{\text{norm}}(\hbar - i0) = +1 \cdot e^{-\frac{4}{3\hbar}} I_2^{\text{norm}}(\hbar), \quad \hbar \in \mathbb{R}_{\geq 0}
\]
\[
I_2^{\text{norm}}(\hbar - i0) - I_2^{\text{norm}}(\hbar + i0) = -1 \cdot e^{+\frac{4}{3\hbar}} I_1^{\text{norm}}(\hbar), \quad \hbar \in \mathbb{R}_{\leq 0}
\]

\text{Jumps:}

\text{modified, of second kind} (Bessel function): \( X = \mathbb{C}^\times, f = x + \frac{1}{x}, \mu = \frac{dx}{x} \). Critical points: \( x = \pm 1 \),
critical values: \( z_1 = -2, z_2 = +2 \). Stokes indices: \( n_{12} = +2, n_{21} = -2 \).

Bessel integral: \( \int_0^\infty e^{-\frac{1}{\hbar}(x + \frac{1}{x})} \frac{dx}{x} = 2K_0\left(\frac{2}{\hbar}\right) \sim \sqrt{\pi \hbar} e^{-\frac{2}{\hbar}} \sum_{n\geq 0} \frac{(-1)^n((2n-1)!!)^2}{n! 16^n} \hbar^n \)
4. Example: Gamma function, infinitely many critical values

Consider algebraic variety \( \mathbb{C}^\times \) with (closed) 1-form \( \eta = (1 - 1/x)dx \) which is not a differential of a function, as it has a non-trivial period \( \oint \eta = -2\pi i \). In order to represent \( \eta \) as differential of a function, we should go to the universal \( \mathbb{Z} \)-cover \( \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \). Denote by \( t \) the coordinate on the cover, so \( x = e^t \). Then we have

\[
\text{(pullback of } \eta \text{)} = dx - \frac{dx}{x} = dx - dt = d(e^t - t), \quad f := e^t - t
\]

Function \( f = f(t) \) has infinitely many critical points \( t_k = 2\pi i k, \quad k \in \mathbb{Z} \).

\(~\sim~\text{critical values: } z_k = 1 - 2\pi i k.\) The normalized integral for \( \Re \hbar > 0, \text{ any } k \in \mathbb{Z} \) is

\[
I_{\bullet \norm}^k(\hbar) := \frac{e^{1/\hbar}}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{(-e^t+t)/\hbar} dt = \frac{e^{1/\hbar}}{\sqrt{2\pi \hbar}} \int_{0}^{\infty} e^{-x/\hbar} x^{1/\hbar} dx = \frac{e^{1/\hbar} \hbar^{1/\hbar} \Gamma(1/\hbar)}{\sqrt{2\pi \hbar}}
\]

For \( \Re \hbar < 0 \) one has \( I_{\bullet \norm}^k(\hbar) = 1/I_{\bullet \norm}^k(-\hbar) \) (does not depend on \( \bullet = k \in \mathbb{Z} \))

Asymptotic expansion: \( I_{\bullet \norm}^k(\hbar) \xrightarrow{\hbar \to 0} 1 + \frac{1}{12} \hbar + \frac{1}{288} \hbar^2 - \frac{139}{51840} \hbar^3 + \ldots \) (Stirling formula).
4. Example: Gamma function, infinitely many critical values

We get basically two functions:

\[
\begin{align*}
I_R(h) &:= \frac{e^{1/h} h^{1/h} \Gamma(1/h)}{\sqrt{2\pi h}} & \text{for } \Re h > 0 \\
I_L(h) &:= 1/I_R(-h) & \text{for } \Re h < 0
\end{align*}
\]

Jump formulas:

\[
\begin{align*}
I_L(h) &= I_R(h) \cdot (1 - \exp(-\frac{2\pi i}{h})) & \text{for } h \in i \mathbb{R}_{>0} \\
I_R(h) &= I_L(h) \cdot (1 - \exp(\frac{2\pi i}{h}))^{-1} & \text{for } h \in i \mathbb{R}_{<0}
\end{align*}
\]

\[\Rightarrow\] Stokes indices \( n_{kk'} = \begin{cases} 
-1 & k' = k - 1 \\
+1 & k' > k \\
0 & \forall k, k' \in \mathbb{Z}
\end{cases} \]

Puzzle: for quantized values of \( h = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \ldots \) function \( e^{-f/h} \) descends to \( \mathbb{C}^\times \)-valued function on \( \mathbb{Z} \)-quotient \( \mathbb{C}^\times \) with coordinate \( x = e^t \neq 0 \). Hence for such \( h \) the contour integral \( \int e^{-f/h} dx/x \) is well-defined. How it is related to the integrals over Lefschetz thimbles? A similar problem in quantum Chern-Simons theory, see later.
5. Example: Infinite-dimensional integral (heat kernel)

Let \((M, g)\) be a real-analytic Riemannian manifold which admits a “reasonable” complexification \((M_C, g_C)\). Fix two points \(p_0, p_1 \in M\), and consider the path integral

\[
I(\hbar) := \int_{\text{Paths } \phi: [0,1] \to M, \phi(0)=p_0, \phi(1)=p_1} e^{-\frac{S(\phi)}{\hbar}} D\phi,
\]

where \(S(\phi) = \frac{1}{2} \int_0^1 \left| \frac{d\phi(t)}{dt} \right|^2 dx\)

The integration domain is as a totally real infinite-dimensional contour \(\gamma\) in \(X := \text{infinite-dimensional complex manifold of paths connecting } p_0 \text{ and } p_1 \text{ in } M_C\). Dirichlet functional \(\phi \in \gamma \mapsto S(\phi) \in \mathbb{R}\) extends to a holomorphic function \(f : X \to \mathbb{C}\), the fictitious ”Lebesgue measure” \(D\phi\) ”extends” to a holomorphic volume form \(\mu\) on \(X\).

**Mathematically rigorous interpretation** (Feynman-Kac formula):

\[
I(\hbar) = \text{heat kernel } K(p_0, p_1; t) = \langle p_0 | e^{-\frac{1}{2}\Delta t} | p_1 \rangle
\]

where time \(t > 0\) (for the Brownian motion) is identified with the Planck constant \(\hbar\).
5. Example: Infinite-dimensional integral (heat kernel)

Short-time asymptotic expansion of the heat kernel:

\[ K(p_0, p_1; t) \overset{t \to +0}{\sim} (2\pi t)^{-\dim M/2} e^{-\frac{\text{dist}(p_0, p_1)^2}{2t}} \sum_{n \geq 0} c_n t^n \]

**Conjecture**: sequence \( c_n \) has factorial growth, is resurgent, and the Borel transform has singularities at numbers \( \ell_\alpha^2/2 \in \mathbb{C} \) where \( \ell_\alpha \) are lengths of complex geodesics connecting \( p_0, p_1 \) (formal solutions of Euler-Lagrange equation for \( \phi : [0, 1] \to (M_\mathbb{C}, g_\mathbb{C}) \)).

**McKean, 1970**

\[ X = \mathbb{H}^2 : \quad K(p_0, p_1; t) = \frac{\sqrt{2} e^{-t/8}}{(2\pi t)^{3/2}} \int_{\ell^2/2}^{\infty} \frac{e^{-z/t}}{(\cosh \sqrt{2}z - \cosh \ell)^{1/2}} \, dz \]

\( \ell := \text{dist}(p_0, p_1) \)

Critical values

\[ z_n = \frac{(\ell + 2\pi in)^2}{2} \]

Also works for spheres, flat tori, compact surfaces with hyperbolic metric, . . .
6. Example: complex Chern-Simons, paths in $K_2$-geometry, sums of dilogs

3 different series of examples, hypothetically give the same class of data $(z_\alpha), (n_{\alpha,\beta})$:

- $X_0 = \mathbb{Z}$-cover of $X'_0 := \{ C^\infty$-connections on G-bundle $\downarrow \}$ /gauge equivalences, where $M$ is an oriented manifold of dimension 3, G is a complex semisimple group.

- $X_1 = \mathbb{Z}$-cover of $X'_1 := \left\{ C^\infty$-paths $\phi : [0, 1] \to (\mathbb{C}^\times)^{2n} \mid \phi(0) \in L_0, \phi(1) \in L_1 \right\}$ where $L_0, L_1$ are algebraic $K_2$-lagrangian submanifolds of $(\mathbb{C}^\times)^{2n}$:  

  $$\sum_{i=1}^{n}[z_i]|_{L_\epsilon} \wedge [z_{n+i}]|_{L_\epsilon} = 0 \in K_2(L_\epsilon), \quad \epsilon = 0, 1 \quad (K_2$-$condition$ \ will$ explain$ later)$$

- $X_3 = \mathbb{Z}$-cover of $X'_3 := \{ \text{C}^\infty$-paths $\phi : [0, 1] \to (\mathbb{C}^\times)^{2n} \mid \phi(0) \in L_0, \phi(1) \in L_1 \}$

where $L_0, L_1$ are algebraic $K_2$-lagrangian submanifolds of $(\mathbb{C}^\times)^{2n}$.

$X_0 = \mathbb{Z}$-cover of $X'_0 := a \mathbb{Z}^m$-cover of $(\mathbb{C}^\times)^m - \bigcup_{\vec{\nu} \in B} \{ \vec{x} \in (\mathbb{C}^\times)^m \mid \vec{x}^{\vec{\nu}} = 1 \}$ where $B \subset \mathbb{Z}^m - 0$ is a finite subset,

$$\vec{x}^{\vec{\nu}} := \prod_{i=1}^{m} x_i^{\nu_i} \text{ for } \vec{x} = (x_1, \ldots, x_m) \in (\mathbb{C}^\times)^m \text{ and } \vec{\nu} = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m.$$
Case of connections on a 3-dimensional manifold $M$:

On $X_3 = \{\text{Connections}\}$ functional $f$ is the Chern-Simons action:

for a $SL(N, \mathbb{C})$-connection $\nabla = d + A$ in the trivialized bundle

$$f(\nabla) := \int_M \text{Tr} \left( \frac{\text{Ad}A}{2} + \frac{A^3}{3} \right), \quad A \in \text{Mat}(N \times N, \Omega^1(M)), \, \text{Tr}(A) = 0$$

Ambiguity under gauge transformations: $f(g^{-1}\nabla g) - f(\nabla) \in (2\pi i)^2\mathbb{Z}$.

For special “quantized” values of $\hbar$:

$$\hbar = \frac{2\pi i}{k}, \, k = \pm 1, \pm 2, \ldots$$

$\leadsto$ well-defined function $\exp(-f/\hbar)$ on $\mathbb{Z}$-quotient $X_3'$. Canonical “compact” cycle of integration $= \{\text{unitary connections}\} \leadsto$ quantum Chern-Simons theory at level $k$ (for $k > 0$).
6. Example: complex Chern-Simons, paths in $K_2$-geometry, sums of dilogs

**Case of paths connecting $K_2$-lagrangian subvarieties $L_0, L_1 \in (\mathbb{C}^\times)^{2n}$:**

Examples of $K_2$-lagrangian subvarieties in $(\mathbb{C}^\times)^{2n}$ endowed with the standard symplectic form $\omega = \sum_{i=1}^{n} d \log(z_i) \wedge d \log(z_{n+i})$:

$$L = \{(z_1, \ldots, z_{2n}) \in (\mathbb{C}^\times)^{2n} \mid \forall i = 1, \ldots, n: \ z_{n+i} = 1 \text{ or } (1 - z_i)\}$$

and its images under $Sp(2n, \mathbb{Z})$-action.

For any $K_2$-lagrangian $L$ and any $\delta \in H_2(((\mathbb{C}^\times)^{2n}, L; \mathbb{Z})$ one has $\int_\delta \omega \in (2\pi i)^2 \mathbb{Z}$.

Functional $f$ on $\{\text{paths connecting two } K_2\text{-lagrangian } L_0, L_1\}$ is defined up to $(2\pi i)^2 \mathbb{Z}$:

$$f(\phi) - f(\phi') = \int_{4\text{-gon}} \omega$$
6. Example: complex Chern-Simons, paths in $K_2$-geometry, sums of dilogs

**Case of sums of dilogarithms:**

**DATA:**

- a finite subset $B \subset \mathbb{Z}^m - 0$
- a collection of integer non-zero weights $w_{\vec{\nu}} \in \mathbb{Z} - 0$ for all $\vec{\nu} \in B$
- an even quadratic form $b = (b_{ij})_{1 \leq i, j \leq m}$, $b_{ij} = b_{ji} \in \mathbb{Z}$, $b_{ii} \in 2\mathbb{Z}$

$\leadsto$ multivalued function

$$f := \sum_{\vec{\nu} \in B} w_{\vec{\nu}} \text{Li}_2(x^{\vec{\nu}}) + \frac{1}{2} \sum_{i,j} b_{ij} \log(x_i) \log(x_j)$$

recall: $\text{Li}_2(x) := \sum_{k \geq 1} \frac{x^k}{k^2}$

Its differential $\eta := df$ is well-defined on $X' := \mathbb{Z}^m$-cover of $(\mathbb{C}^\times)^m - \bigcup_{\vec{\nu} \in B} \{x^{\vec{\nu}} = 1\}$

$$df = \sum_{i=1}^m \left( - \sum_{\vec{\nu} \in B} \nu_i w_{\vec{\nu}} \log(1 - x^{\vec{\nu}}) + \sum_{j=1}^m b_{ij} \log(z_j) \right) d \log(x_i)$$

Periods of $\eta$ belong to $(2\pi i)^2 \mathbb{Z}$, function $f$ is well-defined on a $\mathbb{Z}$-cover $X$ of $X'$. 

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6. Example: complex Chern-Simons, paths in $K_2$-geometry, sums of dilogs

In all 3 situations the set of critical values is a \textit{finite} union of arithmetic progressions in $\mathbb{C}$ each with the step $(2\pi i)^2 = -39.4784\ldots$

\[ \alpha = (\alpha, k) \]
\[ z_\alpha = z_\alpha + (2\pi i)^2 k \]

The reason is that in each of 3 situations, critical points (up to $\mathbb{Z}$-action) are solutions of a system of algebraic equations, with coefficients in $\mathbb{Q}$, of expected dimension 0.

- case 3: representations $\pi_1(M) \to G$ (the group can be defined over $\mathbb{Q}$)
- case 1: intersection $L \cap L'$ of two algebraic subvarieties defined over $\mathbb{Q}$
- case 0: solutions of a system of \textit{algebraic} equations $\exp(x_i \partial_{x_i} f) = 1$, $i = 1, \ldots, m$

\textit{Same critical values, - image of Beilinson-Borel regulator $K_3^{ind}(\mathbb{Q}) \to \mathbb{C}/(2\pi i \mathbb{Z})^2$.}
6. Example: complex Chern-Simons, paths in $K_2$-geometry, sums of dilogs

**Conjecture:** one can identify situations $3, 1, 0$ not only matching the critical values $z_{(\alpha, k)}$, but also the *Stokes indices* $n_{(\alpha, k), (\alpha', k')} =: n_{\alpha, \alpha'; \ell - k}$.

**Analogy:** Stokes indices of an $\infty$-dim. path integral (heat kernel) = those of a finite-dim. exponential integral.

One can effectively study topology in the case $0$, an example:

$$f = \text{Li}_2(x) + \log(x)^2 \sim X'_0 = \{(x, t) \in \mathbb{C}^2 | \frac{x^2}{1-x} = e^t\} \iff x = \frac{-e^t \pm e^{t/2} \sqrt{e^t + 4}}{2}$$

$\infty$ genus hyperelliptic curve

Map $\exp(\frac{f}{2\pi i}) : X'_0 \to \mathbb{C}^\times$ is ramified at a finite set of points in $\mathbb{C}^\times$, monodromy is accessible.

The angle-ordered product $\overset{\triangleright}{\prod}$ of $\infty$ many Stokes matrices $S_\theta$ can be identified with the “monodromy” of a $q$-difference equation where

$$q = e^{-\frac{(2\pi i)^2}{\hbar}}$$

replace by one cut in RH problem